Analytically heavy spaces: analytic Cantor and analytic Baire theorems

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Abstract

Motivated by recent work, we establish the Baire Theorem in the broad context afforded by weak forms of completeness implied by analyticity and $\mathcal{K}$-analyticity, thereby adding to the 'Baire space recognition literature' (cf. [AL], [HM]). We extend a metric result of van Mill, obtaining a generalization of Oxtoby's weak α-favourability conditions (and therefrom variants of the Baire Theorem), in a form in which the principal role is played by $\mathcal{K}$-analytic (in particular analytic) sets that are 'heavy' (everywhere large in the sense of some σ-ideal). From this perspective fine-topology versions are derived, allowing a unified view of the Baire Theorem which embraces classical as well as generalized Gandy-Harrington topologies (including the Ellentuck topology), and also various separation theorems. A multiple-target form of the Choquet Banach-Mazur game is a primary tool, the key to which is a restatement of the Cantor theorem, again in $\mathcal{K}$-analytic form.

Keywords: analytic, K-analytic, analytically heavy, weakly α-favourable, heavy sets, irreducible submap, Cantor Theorem, Baire space, Banach-Mazur games, Choquet games, Luzin separation, fine topology, density topology, Gandy-Harrington topology, Ellentuck topology, O'Malley topologies, Effros Theorem.

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1 Analytic Baire & Analytic Cantor Theorems: the ABC of localization

We consider games of Banach-Mazur and Choquet type in the category of $\mathcal{K}$-analytic spaces (here $\mathcal{K}$ refers to compact subspaces, all terms to be defined below), a category of Lindelöf spaces broader than the complete spaces, but smaller and simpler than the category of p-spaces, with which they share many features. (Actually, the current approach has non-Lindelöf generalizations – see no further than the end of this paragraph.) In doing so we demonstrate why analytic spaces provide a satisfying, viable replacement for Polish spaces in circumstances where completeness is lacking. (A similar theme is developed for measure theory by Fremlin, for which see [Fre-4] Part I, Ch. 43) Indeed, completeness will be absent, even in a compact metric context, from a respectably-defined subspace, when it turns out to have descriptive character more complex than $\mathcal{G}_\delta$. See §5 for examples from recent literature of established theorems whose range of applicability has been considerably broadened by an analytic-space hypothesis in lieu of completeness (in particular the Effros Open Mapping Principle), and the more recent results in [Ost-LBIII] on infinite combinatorics (concerning shift-compactness and exhibiting measure-category duality), and [Ost-Joint] (separable case) and [Ost-AB] (for the non-separable case) on continuity in groups. We briefly consider the metric-space category of non-separable classical analytic spaces, which viewed through their 'extended Souslin' representation share some of the properties of $\mathcal{K}$-analytic spaces developed here. As the technical apparatus there is more demanding, we limit ourselves mostly to informative comment. We again refer to [Ost-AB] for applications in the non-separable metric context of the approach here.

An observation of van Mill in the metric setting of [vM] (Prop. 2.2) is our point of departure, given below as Theorem A – in the context of Hausdorff spaces. Here it is recognized as an equivalent analytic-sets restatement of the standard Baire Theorem (i.e., *in a Polish space, the intersection of dense open sets is dense),* with the advantages of symmetry: analytic sets replace both the space of the premise (that the space is Polish), and the intersection set of the conclusion, and with a weaker hypothesis of largeness, since 'dense *open*' is replaced throughout by 'dense and everywhere *non-meagre and analytic*'. Passage to other σ-ideals immediately suggests itself. (Note that open sets might not be analytic in a more general context – see Lemma 1.)

Indeed, by working with various σ-ideals $\mathcal{I}$ on $\mathcal{K}$-analytic sets, we are also able to establish several analogues of Theorem A for the contexts of: measure theory, Luzin-separation theory (with Loveau's separation theorem in its ambit, see §3.2), and fine topology, including the Baire Theorem satisfied by the Gandy-Harrington topology from mathematical logic. (The latter is a refinement of the metric topology on $\mathbb{N}^{\mathbb{N}}$, generated by some Σ_1^1-sets, i.e. again certain analytic sets – see §2.)

Our arguments rely on two ingredients:

(i) *Localization* with respect to $\mathcal{I}$, and

(ii) *Analytic Cantor Theorems* (variants of 'sieve completeness', but in an instantly recognizable form).

It is precisely the analytic Cantor argument that may be viewed as constructing strategies in a topological Banach-Mazur game, yielding a reinterpretation that $\mathcal{I}$-heavy (definition below) analytic spaces are 'favourable' (to one player of the game, by yielding up a winning strategy).

Van Mill's variant and proof of Theorem A (in the metric context) refer to analytic sets only to obtain an elementary proof of the Effros theorem that avoids use of the Baire property (so avoids a feature common to several earlier proofs – see §5). In fact Theorem A follows easily (see below) from an appeal to Nikodym's Theorem on the preservation of the Baire property by the Souslin operation (see [Jay-Rog] p. 42-43), and the fact that closed sets have the Baire property. Our new viewpoint enables us to extract more from van Mill's approach. In particular, it enables, as we show, the construction of strategies in certain topological Banach-Mazur games, when the spaces are continuous images of a Lindelöf, Čech-complete space. (But see [Kech] 21.6 for a Banach-Mazur game proof that classical analytic sets have the Baire property.)

The focus on analytic spaces is motivated by a desire to unify some 'mirroring themes', recurring justifiably in two literatures: classical analysis (such as the Effros theorem) and mathematical logic – see for instance Becker's recent quest for a new definition of 'satisfactory' analyticity in [Bec]. A brief separate section (§5.1) points to the broader picture, obtained by replacing $\mathcal{K}$-analytic spaces by p-spaces.

1.1 Measure-category duality & Analytic Baire Theorems

Notation. Write I for $\mathbb{N}^{\mathbb{N}}$ endowed with the product topology (treating $\mathbb{N}$ as discrete) and $i|n := (i_1, ..., i_n)$ for $i \in I$. Denote by $\mathcal{K} = \mathcal{K}(X)$ the family of compact subsets of a topological space X, $\mathcal{G} = \mathcal{G}(X)$ the open sets, and $\mathcal{F} = \mathcal{F}(X)$ the closed sets.

Definitions 1 ($\mathcal{K}$-analytic spaces). We recall from [Jay-Rog] that for X a Hausdorff space, a map $K : I \to \wp(X)$ is *compact-valued* if $K(i)$ is compact for each $i \in I$, and *singleton-valued* if each $K(i)$ is a singleton. K is *upper-semicontinuous* if, for each $i \in I$ and each open U in X with $K(i) \subseteq U$, there is n such that $K(i') \subseteq U$ for each i' with $i'|n = i|n$.

A subset in X is $\mathcal{K}$-*analytic* if it may be represented in the form $K(I)$ for some compact-valued, upper-semicontinuous map $K : I \to \wp(X)$. We say that X is a $\mathcal{K}$-*analytic space* if X itself is a $\mathcal{K}$-analytic set.

Notation (continued). Recalling the notation $i|n := (i_1, ..., i_n)$, we put $I(i|n) := \{i' : i'|n = i|n\}$ and $I_<(i|n) := \{i' \in I : i'_m \leq i_m$ for $m \leq n\}$. So $I_<(i) := \bigcap_n I_<(i|n)$ is compact (by Tychonoff's Theorem). Correspondingly we write $K(i|n) := K(I(i|n)) = \bigcup\{K(i') : i'|n = i|n\}$, $K_<(i|n) := K(I_<(i|n))$, and $K_<(i) = K(I_<(i))$; the latter is also compact if K is compact-valued and upper-semicontinuous.

We recall that a set A in X is obtained from a family $\mathcal{H}$ of subsets of X by the *Souslin operation* (briefly is *Souslin-*$\mathcal{H}$), if there is a *determining system* $H = \langle H(i|n)\rangle$ assigning to each $i|n$ a set $H(i|n) \in \mathcal{H}$ with

$$A = \bigcup_{i\in I} \bigcap_{n\in\omega} H(i|n).$$

Definition 2. If $K : I \to \mathcal{K}(X)$ is upper semi-continuous, say that K is $\mathcal{H}$-*circumscribed* if there is a determining system $\langle H(i|n)\rangle$ assigning to each $i|n$ a set $H(i|n) \in \mathcal{H}$ with $K(i) = H(i) := \bigcap_{n\in\omega} H(i|n)$ such that $H(i) \subseteq U$ for U open implies that $H(i|n) \subseteq U$ for some n.

Remarks. 1. If K is upper-semicontinuous, as above, and X is Hausdorff, then

$$K(I) = \bigcup\nolimits_{i \in I} \bigcap\nolimits_{n \in \omega} \mathrm{cl} K(i|n),$$

so K is $\mathcal{F}$-*circumscribed*, in particular a $\mathcal{K}$-analytic set is Souslin-$\mathcal{F}$. Indeed, as $K(i) \subseteq \bigcap_{n \in \omega} \mathrm{cl}K(i|n)$, the inclusion from left to right is clear; for the other direction, if $x \in (\bigcap_{n\in\omega} \mathrm{cl}K(i|n)) \backslash K(i)$ for some i, then there is U open with $x \notin \mathrm{cl}U$ and $K(i) \subseteq U$, and so $K(i|n) \subseteq U$ for some n, yielding the contradiction that $x \notin \mathrm{cl}K(i|n) \subseteq \mathrm{cl}U$.

2. If further X is a metric space and $G(i|n) := B_{1/n}(K(i|n)) = \{x : d(x,y) < 1/n$ for some $y \in K(i|n)\}$, then $K(I) = \bigcup_{i\in I} \bigcap_{n\in\omega} G(i|n)$ is a Souslin-$\mathcal{G}$ representation such that:

(i) $\bigcap_{n\in\omega} G(i|n) = K(i)$ is compact;

(ii) $K(i) \subseteq U$, for U open, implies that $G(i|n) \subseteq U$ for some n.

Then K is $\mathcal{G}$-*circumscribed*. Indeed, if $K(i) \subseteq U$, then by compactness there is n such that $B_{2/n}(K(i)) \subseteq U$. But for $m > n$ large enough $K(i|m) \subseteq B_{1/n}(K(i))$, and so $B_{1/m}(K(i|m)) \subseteq B_{1/m}(B_{1/n}(K(i))) \subseteq B_{2/n}(K(i)) \subseteq U$.

We will return to this observation when we consider p-spaces in §5.1.

3. See §1.3 for generalizations of the above in which I is replaced by $J = \kappa^{\mathbb{N}}$ with κ an arbitrary infinite cardinal.

Definition (analytic spaces). Call a Hausdorff space X *analytic* if X is the continuous image of a closed subset of I, or equivalently of a Polish space. So an analytic space is $\mathcal{K}$-analytic, since singletons are compact; a $\mathcal{K}$-analytic subset of an analytic set is analytic. A $\mathcal{K}$-analytic space is more general, since it is a continuous image of a Lindelöf Čech-complete space (by Jayne's Theorem – [Jay-Rog] Th.2.8.1; cf. Hansell [Han-92] Th. 3.1).

Note that if X is metric and $\mathcal{K}$-analytic, then without loss of generality we may arrange to have $\mathrm{diam}_X(K(i|n)) < 2^{-n}$. This implies that $K(i) = \{k(i)\}$ on a closed subset of I on which k is continuous. See [Jay-Rog] §2.8 for background on different definitions of analyticity, and §2.11 for applications to Banach spaces under the weak topology.

In a metric analytic space open sets, being $\mathcal{F}_\sigma$, are analytic. Note that an open subset of a Polish space is again a Polish space. This yields the following generalizations.

Lemma 1. (i) *In an analytic space, all open sets are analytic.*

(ii) *A regular $\mathcal{K}$-analytic space (in particular a regular analytic space) is Lindelöf, so open sets are $\mathcal{K}$-analytic iff they are $\mathcal{F}_\sigma$.*

Proof. (i) If X is the continuous image of I under α, then each open U in X is the continuous image under α of $\alpha^{-1}(U)$, an open subset in I.

(ii) By regularity, for U open, U has an open covering by sets V with $\mathrm{cl}V \subseteq U$. If U is $\mathcal{K}$-analytic, it is Lindelöf, and so U has a countable covering by open sets V_n with $\mathrm{cl}V_n \subseteq U$; so $U = \bigcup_n \mathrm{cl}V_n$ is an $\mathcal{F}_\sigma$. For the converse, as Souslin-$\mathcal{F}_\sigma$ subsets of a $\mathcal{K}$-analytic space are $\mathcal{K}$-analytic, open sets are $\mathcal{K}$-analytic if they are $\mathcal{F}_\sigma$. $\square$

Remark. 1. See §3 for a useful assumption weaker than that open sets are $\mathcal{K}$-analytic. Of course for U open, $(\mathrm{cl}U)\backslash U$ is closed nowhere dense, so modulo a meagre set an open set is closed, so 'almost $\mathcal{K}$-analytic'.

2. The two parts of Lemma 1 are close, since ([Levi], [HrAv]) every analytic Baire space has a dense completely metrizable subspace; in fact every regular analytic space has a finer topology in which it is metrizable (just declare the countably many set $\mathrm{cl}\alpha(I(i|n))$ to be open). The former condition is related to the existence of winning strategies in certain topological games (see Proposition L3 in §3 and [Wh] Th. (11)).

Definitions 3 ($\mathcal{I}$-Heavy parts). Let X be a topological space and $\mathcal{I}$ a σ-ideal in $\wp(X)$. We have in mind $\mathcal{I} = \mathcal{M}$, the meagre sets, $\mathcal{N}$, the null sets (if X carries a measure), and the trivial ideal $\mathcal{I} = \{\emptyset\}$. We will also consider for A a fixed (usually analytic) set, with $\mathcal{B}o(X)$ denoting the σ-algebra of Borel subsets, the *omission* σ-ideal $\mathcal{I}_A := \{C : (\exists B \in \mathcal{B}o(X))[C \subseteq B$ and $B \cap A = \emptyset]\}$.

Say that $\mathcal{I}$ has the *Borel envelope property* if for any analytic $A \in \mathcal{I}$ there is a Borel $B \in \mathcal{I}$ with $A \subseteq B$. All four of $\mathcal{M}, \mathcal{N}$, $\mathcal{I}_A$ and $\{\emptyset\}$ have this property. (The Borel set can be $\mathcal{F}_\sigma$ in the first case, and $\mathcal{G}_\delta$ in the second case.)

Say that S is $\mathcal{I}$*-heavy*, resp. $\mathcal{I}$*-heavy on* G, in X if $S \cap U \notin \mathcal{I}$ for every open set U in X meeting S, resp. for every open U meeting $S \cap G$. (The term 'heavy' is established, going back to [BrGo]; van Mill [vM] calls a dense, heavy set 'fat'. See also [St1] for a general 'kernel' approach.) We will drop the reference to $\mathcal{I}$ whenever convenient, when context allows.

For $\mathcal{I} = \{\emptyset\}$, the trivial ideal, any set is $\mathcal{I}$-heavy (vacuously so, when empty).

In a space X, the $\mathcal{I}$*-light part* of A is defined to be the set $L_\mathcal{I}(A) := \bigcup\{V \cap A : V \text{ open and } A \cap V \in \mathcal{I}\}$. The *heavy part* of A is the complementary set $H_\mathcal{I}(A) = A \backslash L_\mathcal{I}(A)$. We say that A is *heavy* (*heavy* on G) if $L_\mathcal{I}(A)$ is empty (if $L_\mathcal{I}(A) \cap G$ is empty).

Definition. Say that $\mathcal{I}$ has the *localization property* if $L_\mathcal{I}(A) \in \mathcal{I}$ for each A.

This 'light' part property, when true, has implications for the heavy part. In §3.2 we establish a 'heavy' property without a 'light' one being available. (For relevance of light localization in fine topologies, to ideal base operators, see [LMZ] §1.C, where also counterexamples arising in harmonic spaces are cited.)

The next result is *Banach's localization principle* (for which see [Jay-Rog] p. 42, [Kel] Th. 6.35, [Oxt2] Ch. 16 under the name of Banach's Category Theorem, or [Kur-1] §10.III under the name Union theorem).

Category Localization Lemma (Light version). *$L_\mathcal{M}(A)$ is meagre for each set A.*

We next verify directly that the measure analogue also holds. That case may be deduced from the Category Localization Lemma using the existence of the density topology, $\mathcal{D}$. To do this we recall the definition of $\mathcal{D}$ for the case of $\mathbb{R}$, and the general case of a metrizable topological group. (For this and especially generalizations involving other σ-ideals, see [Wil].)

In the case of $\mathbb{R}$, with $|.|$ Lebesgue measure, $\mathcal{D}$, the density topology, comprises measurable subsets D such that all points of D are density points of D. It is precisely because $\mathcal{D}$ is closed under arbitrary unions (see [GNN], [GoWa] and [Mar]) that $\mathcal{D}$ is a topology. In any locally compact metrizable topological group this idea may be repeated verbatim with $|.|$ the Haar (invariant) measure, by appeal to a combination of the results of Martin [Mar] and Mueller [Mue]. (See e.g. [BOst-N] for an exposition, or [Ost-LBIII] for the more general normed group case.) For $\mathcal{I} = \mathcal{N}$, the null sets, and A arbitrary, $L_{\mathcal{N}}(A) := \bigcup\{A \cap D : D \in \mathcal{D} \text{ and } A \cap D \in \mathcal{N}\}$; however, we are interested only in A measurable.

We write A^* for the set of those points of A that are density points of A. This is a measurable set and comprises almost all points of A, by the Lebesgue Density Theorem in the Lebesgue case, and by Mueller's result in the Haar case. So for any A, one has $A^* = \mathrm{int}_{\mathcal{D}}(A)$, the $\mathcal{D}$-interior of A, and also $A^* = A \cap \mathrm{int}_{\mathcal{D}}\mathrm{cl}_{\mathcal{D}}(A)$. (To see this observe first that $\mathrm{cl}_{\mathcal{D}}(A)$ comprises all the non-dispersion points of A, i.e. including any not in A, and only the density points are in the interior. Indeed, if x is a dispersion point of A, then x is a density point of $(X\backslash A)$ and so $(X\backslash A)^*$ is a $\mathcal{D}$-neighbourhood of x disjoint from A; conversely, if $x \notin \mathrm{cl}_{\mathcal{D}}(A)$, then there is D disjoint from A with $x \in D \in \mathcal{D}$ and so x is a dispersion point of A. Finally, $\mathrm{int}_{\mathcal{D}}\mathrm{cl}_{\mathcal{D}}(A) = (\mathrm{cl}_{\mathcal{D}}(A))^*$.)

Measure Localization Lemma (Light version). *For A measurable, $L_{\mathcal{N}}(A) \in \mathcal{N}$.*

Proof. Since $N := L_{\mathcal{N}}(A) = A \cap \bigcup\{D : D \in \mathcal{D} \text{ and } A \cap D \in \mathcal{N}\}$ and $\bigcup\{D : D \in \mathcal{D} \text{ and } A \cap D \in \mathcal{N}\}$ is measurable, the set N is measurable. Let N^* be the set of density points of N; then $N^* \subseteq N \subseteq A$, and by Lebesgue's (or Mueller's) density theorem $|N^*\backslash N| = 0$. If $|N^*| > 0$, then $N^* \in \mathcal{D}$. Pick any $a \in N^*$; then, as $N^* \subseteq N$, there is a $D \in \mathcal{D}$ with $a \in D$ and $D \cap A \in \mathcal{N}$. But $a \in N^* \cap D$, so $0 < |N^* \cap D| \leq |A \cap D| = 0$, a contradiction. So $|N^*| = 0$, and so $|N| = 0$, as asserted. $\square$

Lemma 2. *If a $\mathcal{K}$-analytic A is dense and $\mathcal{M}$-heavy in the Hausdorff* *space X, then A is co-meagre and dense, and so X is a Baire space.*

Proof. As X is Hausdorff, any $\mathcal{K}$-analytic set is Souslin-$\mathcal{F}$ (see above). Any closed set has the Baire property (as $F\backslash\mathrm{int}(F)$ is nowhere dense), so A has the Baire property by Nikodym's Theorem on the preservation of the Baire property by the Souslin operation (see again [Jay-Rog] p. 42-43). Put $A = (V\backslash M) \cup N$ with M, N in $\mathcal{M}$ and V open in X. As A is dense, for W open the set $A \cap W$ is non-empty, and so also not meagre (as A is heavy). So $W \cap V \supseteq (W \cap A)\backslash N$ is non-empty. So V is dense, and so A is co-meagre. Since $A \cap W$ is non-meagre for each non-empty open set W, the set $V \cap W$ is also non-meagre, and so X is Baire. □

With minor adjustments, the argument transfers to the measure case. In fact one need only know that $\mathcal{D}$ is a refinement of the original topology so that a set has the Baire property iff it is measurable, and is meagre iff it is null (cf. [Kech] Ex. 17.47 in the case of $\mathbb{R}$, or [Mar]). To obtain some more information, we repeat this short argument. For the condition that open sets be $\mathcal{K}$-analytic, refer to Lemma 1.

Lemma 2′. *In a Hausdorff space $(X, \mathcal{T})$ possessing a density topology $\mathcal{D}$ refining $\mathcal{T}$, if a $\mathcal{K}$-analytic set A is $\mathcal{D}$-dense and $\mathcal{N}$-heavy, then A is $\mathcal{T}$-dense and co-null, and X is both a Baire space under $\mathcal{D}$, and modulo a null set also Baire under $\mathcal{T}$. Moreover, on the complement in X of a null set, A is $\mathcal{T}$-dense and $\mathcal{M}(\mathcal{T})$-heavy.*

Proof. Any closed set is measurable (since $\mathcal{D}$ refines $\mathcal{T}$), so by the Luzin-Sierpiński Theorem (see e.g. [Jay-Rog] p.42-43) A, being Souslin-$\mathcal{F}$ in X, is measurable. By inner regularity of the measure we may put $A := V\backslash N$ with N in $\mathcal{N}$ and V a $\mathcal{G}_\delta$ under $\mathcal{T}$. As A is $\mathcal{D}$-dense, for W a $\mathcal{D}$-open set $A \cap W$ is non-empty, and so also non-null (as A is heavy). So $W \cap V \supseteq (W \cap A)\backslash N$ is non-empty. So V is $\mathcal{D}$-dense, hence $\mathcal{D}$-co-meagre, so co-null and non-null. Since $A \cap W$ is non-null for each non-empty $\mathcal{D}$-open set W, the set $V \cap W$ is non-null also for W a $\mathcal{T}$-open set, i.e. $V \cap W$ is a dense $\mathcal{G}_\delta$ in W, and so X is $\mathcal{T}$-Baire.

If H is a null $\mathcal{G}_\delta$ with $H \supseteq N$, then $A' := V\backslash H = A\backslash H$ and so A' is $\mathcal{K}$-analytic (being an $\mathcal{F}_\sigma$ in the $\mathcal{K}$-analytic set A); furthermore, A' is dense and heavy on $X\backslash H$. $\square$

Theorem A below is substantially due to van Mill [vM]; cf. Levi [Levi] Th. 2,3. We note below its dual.

Theorem A (Analytic Baire Theorem – Category-heavy, [vM] Prop. 2.2). *In a Hausdorff space X, if A_n are $\mathcal{K}$-analytic, $\mathcal{M}$-heavy and dense in X, then $\bigcap_n A_n \neq \emptyset$.*

Proof of Theorem A. By Lemma 2 X is Baire and each A_n is, modulo a meagre set, dense open. So $\bigcap_n A_n \neq \emptyset$. $\square$

Proposition A (Analytic Baire Theorem – Measure-heavy). *In a Hausdorff space X, if A_n are $\mathcal{K}$-analytic, $\mathcal{N}$-heavy and $\mathcal{D}$-dense in X, then $\bigcap_n A_n \neq \emptyset$.*

Proof of Proposition A. By Lemma 2′ X is non-null and each A_n is co-null. So $\bigcap_n A_n \neq \emptyset$. $\square$

Remarks. We know from Lemma 1 that in a Polish space, and more generally in an analytic metric space, open sets are analytic, so again by Lemma 2 Theorem A restricted to analytic metric spaces is equivalent to the classical (metric) Baire Theorem. The latter is standardly proved using Cantor's (nested sets) Theorem; in the next subsection we derive an analytic Cantor Theorem and later in §3 combine it with van Mill's approach to prove in Theorem 5 a stronger analytic formulation of the Baire Theorem for Hausdorff spaces. Before that, however, in §2 we will see why van Mill's Theorem may be viewed as a general topological version of the Gandy-Harrington Theorem asserting that the Gandy-Harrington topology and other related topologies are Baire. In §4 we conduct a similarly motivated further analysis but now in the context of topologies that refine a metrizable topology.

Convention. Below, the new results are numbered theorems. Lettered theorems, such as Theorem A above, denote attributed results (possibly in more abstract forms than originally formulated). Hybrid results, arising from a new or unifying perspective, appear as Propositions – for instance Proposition A just proved.

1.2 Analytic Cantor Theorems

The following result is implicit in a number of situations, and goes back to Frolík's characterization of Čech-complete spaces as $\mathcal{G}_\delta$ in some compactification ([Frol-60]; see [Eng] §3.9); it may be used to lift theorems about Polish spaces to results on analytic metric spaces and to characterize analytic sets. For example, Frolík in [Frol-70] Th. 2 characterizes analytic sets as intersections of a $\mathcal{G}_\delta$ and a set that is Souslin in its Stone-Čech compactification; in similar spirit Fremlin [Fre] develops the theory of Čech-analytic sets (cf. also [HJR]). In the opposite direction Aarts et al. in [AdGMcD1] and [AdGMcD2] use similar machinery to characterize completeness via compactness. Recall that Cantor's Theorem on the intersection of a nested sequence of closed (or compact, as appropriate) sets has two formulations: (i) referring to vanishing diameters (in a complete-space setting), and (ii) to (countable) compactness. In the spirit of these, we now give two topological versions. We refer to §1.1 for notation.

Theorem $1_{\mathbf{C}}$ (Analytic Cantor Theorem). *Let X be a Hausdorff space and $A = K(I)$ be $\mathcal{K}$-analytic in X, with K compact-valued and upper-semicontinuous.*

If F_n is a decreasing sequence of (non-empty) closed sets in X such that $F_n \cap K(I(i_1, ..., i_n)) \neq \emptyset$, for some $i = (i_1, ...) \in I$ and each n, then $K(i) \cap \bigcap_n F_n \neq \emptyset$.

Equivalently, if there are open sets V_n in I with $\mathrm{cl}V_{n+1} \subseteq V_n$ and $\mathrm{diam}_I V_n \downarrow 0$ such that $F_n \cap K(V_n) \neq \emptyset$, for each n, then

(i) $\bigcap_n \mathrm{cl}V_n$ *is a singleton, $\{i\}$ say;*

(ii) $K(i) \cap \bigcap_n F_n \neq \emptyset$.

Proof. If not, then $\bigcap_n K(i) \cap F_n = \emptyset$ and so, by compactness, $K(i) \cap F_p =$

$\emptyset$ for some p, i.e. $K(i) \subseteq X \backslash F_p$. So by semicontinuity $F_p \cap K(I(i_1, ..., i_n)) = \emptyset$ for some $n \geq p$, yielding the contradiction $F_n \cap K(I(i_1, ..., i_n)) = \emptyset$. □

Theorem 1 has the following filterbase (finite intersection property, 'fip') generalization.

Theorem $1_{\mathbf{Tr}}$ (Analytic Compactness Theorem – Trace). *In the setting of Theorem 1_{C}, for $\mathcal{H}$ a filter base, if for some i and each $n \in \omega$, $H_0 \in \mathcal{H}$ there is $m > n$ and $H \in \mathcal{H}$ with $H \subseteq H_0$ meeting $K(I(i_1, ..., i_m))$, then*

$$K(i) \cap \bigcap \{\mathrm{cl} H : H \in \mathcal{H}\} \neq \emptyset.$$

Proof. If not, and $\emptyset = K(i) \cap \bigcap \{\mathrm{cl} H : H \in \mathcal{H}\}$, then for some finite subfamily $\mathcal{H}'$ we have $\emptyset = K(i) \cap \bigcap \{\mathrm{cl} H : H \in \mathcal{H}'\}$, and so $K(i) \subseteq \bigcup \{X \backslash \mathrm{cl} H : H \in \mathcal{H}'\}$. By upper-semicontinuity, $K(I(i_1, ..., i_n)) \subseteq \bigcup \{X \backslash \mathrm{cl} H : H \in \mathcal{H}'\}$, for some n. As $\mathcal{H}$ is a filter base, there exists $H_0 \in \mathcal{H}$ with $H_0 \subseteq \bigcap \{H : H \in \mathcal{H}'\}$. Now for some $m > n$ and some $H_1 \in \mathcal{H}$ with $H_1 \subseteq H_0$, the set $K(I(i_1, ..., i_m))$ meets H_1, contradicting the fact that $\emptyset = K(i) \cap \bigcap \{H : H \in \mathcal{H}\}$. □

The filter-base version is usually rendered employing 'inclusion' as below (suggesting the shrinking 'diameters' of Cauchy's criterion, ultimately the inspiration of Frolík's [Frol-60]; cf. again [Eng] §3.9 and Hansell [Han-92], §3, p. 281) rather than the 'trace' property above. This has a similar but simpler proof, given below for the sake of completeness. In fact the inclusion version implies the trace version (see the Remark below). In §3.1 we see their duals in the Banach-Mazur 'inclusion' games and the Choquet 'trace' games.

Theorem $1_{\mathbf{Inc}}$ (Analytic Compactness Theorem – Inclusion). *In the setting of Theorem 1_{C}, for $\mathcal{H}$ a filter base, if for some i each $K(I(i_1, ..., i_n))$ contains a member of $\mathcal{H}$, then*

$$\emptyset \neq \bigcap \{\mathrm{cl} H : H \in \mathcal{H}\} \subseteq K(i).$$

Proof. The inclusion is clear. If $\emptyset = K(i) \cap \bigcap \{\mathrm{cl} H : H \in \mathcal{H}\}$, then for some finite subfamily $\mathcal{H}'$ we have $\emptyset = K(i) \cap \bigcap \{\mathrm{cl} H : H \in \mathcal{H}'\}$, and so

$K(i) \subseteq \bigcup\{X\backslash \mathrm{cl}H : H \in \mathcal{H}'\}$. By upper-semicontinuity, $K(I(i_1, ..., i_n)) \subseteq \bigcup\{X\backslash \mathrm{cl}H : H \in \mathcal{H}'\}$, for some n. But $K(I(i_1, ..., i_n)) \supseteq H_0$ for some (non-empty!) $H_0 \in \mathcal{H}$, and so $\emptyset = H_0 \cap H'$ for each $H' \in \mathcal{H}'$, contradicting the fact that $\mathcal{H}$ is a filter sub-base, unless $\mathcal{H}' = \emptyset$. But then $K(i) = \emptyset \supseteq K(I(i_1, ..., i_n)) \supseteq H_0$, giving a final contradiction. □

Remark. To see why Theorem 1_{Incl} implies 1_{Tr}, consider a filter base $\mathcal{H}$ with the trace property of Theorem 1_{Tr} relative to $i \in I$. One may pick for each $H_0 \in \mathcal{H}$ and $n \in \mathbb{N}$ an integer $m = m(n) > n$ and a set $H = H^n(H_0) \subseteq H_0$ in $\mathcal{H}$ such that $H_i^n(H_0) := H \cap K(i_1, ..., i_{m(n)}) \neq \emptyset$. Then $\{H_i^n(H_0) : n \in \mathbb{N},\ H_0 \in \mathcal{H}\}$ is a filter sub-base satisfying the hypothesis of Theorem 1_{Incl} and so

$$\emptyset \neq K(i) \cap \bigcap\{\mathrm{cl}H_i^n(H_0) : n \in \mathbb{N},\ H_0 \in \mathcal{H}\} \subseteq K(i) \cap \bigcap\{\mathrm{cl}H_0 : H_0 \in \mathcal{H}\}.$$

A similar argument can be conducted with a weaker hypothesis by exploiting the compactness of $K_<(i)$ (defined in §1.1).

Theorem $\mathbf{1_{Cpt}}$. *In the setting of Theorem 1_{C}, suppose now that the nested sequence F_n satisfies $F_n \cap K(I_<(i_1, ..., i_n)) \neq \emptyset$, for some $i = (i_1, ...) \in I$ and each n. Then $K_<(i) \cap \bigcap_n F_n \neq \emptyset$.*

Equivalently, if there are open sets V_n in I with $H := \bigcap_n \mathrm{cl}V_n$ non-empty compact such that $F_n \cap K(V_n) \neq \emptyset$ for each n, then $K(H) \cap \bigcap_n F_n \neq \emptyset$.

Proof. If not, then $K_<(i) \cap \bigcap_n F_n = \emptyset$. Since $K_<(i)$ is compact, $K_<(i) \cap F_p = \emptyset$, for some p. By upper-semicontinuity, for each $j \in I_<(i)$ there is $n(j)$ such that $K(j|n(j)) \subseteq X\backslash F_p$. Since $I_<(i)$ is compact, there are $j(1), ..., j(t)$ in $I_<(i)$ and integers $n_s = n(j(s))$ such that $\{I(j(s)|n_s) : s = 1, ..., t\}$ is a finite open covering of $I_<(i)$. Put $q = p + \max_{s \leq t} n_s$.

For $j \in I_<(i|q)$, consider j' with $j'|q = j|q$ and $j' \in I_<(i)$. (For instance, take $j' = j_1, ..., j_q, i_{q+1}, i_{q+2}, ...$) Refer to the finite covering to find s with $j'|n_s = j(s)|n_s$. Then $K(j|q) \subseteq K(j(s)|n_s) \subseteq X\backslash F_p$. So $K(I_<(i|q)) \subseteq X\backslash F_p$, and in particular $K(I_<(i|q)) \cap F_q = \emptyset$, a contradiction. □

The following generalization of Th. 1 is at the heart of both the proof of the Gandy-Harrington Theorem (see §2.2) and likewise of van Mill's proof of the Analytic Baire Theorem, Th. A above.

Theorem 2 (Multiple Analytic Targets - Trace Theorem). *Let X be a Hausdorff space and $A_n = K_n(I)$ be $\mathcal{K}$-analytic in X, with K_n taking singleton or empty values and upper-semicontinuous.*

If F_n is a decreasing sequence of (non-empty) closed sets in X such that

$$F_n \cap \bigcap_{m \leq n} K_m(I(i_1, ..., i_n)) \neq \emptyset,$$

for some $i = (i_1, ...) \in I$ and each n, then $\bigcap_n F_n \cap \bigcap_n A_n(i) \neq \emptyset$.

Proof. If not, write $H_n := K_n(i)$ and $K_n(i) := \{x_n\}$ (whenever $K_n(i)$ is non-empty). By compactness, since $\bigcap_n (F_n \cap H_n) = \emptyset$, there is p with $F_p \cap \bigcap_{n \leq p} H_n = \emptyset$. If $x_m \notin F_p$ or $H_m = \emptyset$ for some $m \leq p$, then $K_m(i) \subseteq X \backslash F_p$, and so $F_p \cap K_m(I_n(i|n)) = \emptyset$ for some $n > p+m$. Then $F_n \cap K_m(I_n(i|n)) = \emptyset$, a contradiction. So $x_m \in F_p$ for all $m \leq p$. Since $F_p \cap \bigcap_{n \leq p} H_n = \emptyset$, for some m, m' we have $x_m \neq x_{m'}$. As X is Hausdorff, for some disjoint U, V we have $x_m \in U$ and $x_{m'} \in V$. So for some $n > m + m' + p$ we have $K_m(i|n) \subseteq U$ and $K_{m'}(i|n) \subseteq V$. So $F_n \cap K_m(i|n) \cap K_{m'}(i|n) = \emptyset$, a contradiction. $\square$

We generalize the last result beyond the singleton-valued to the compact-valued case, which needs a separation lemma. (This generalizes the well-known result that in a Hausdorff space disjoint compact sets may be separated by disjoint open sets – cf. Kelly, [Kel] Th. 5.9.) The next result is shown for regular Hausdorff spaces in [DJRO] (in the course of a proof of Th. 1 there). We work in subspaces that are $\mathcal{K}$-analytic, so the following more intuitive proof applies. Recall from Lemma 1 that an $\mathcal{K}$-analytic space A is Lindelöf and that a regular Lindelöf space is normal (cf. Kelly, [Kel] Lemma 4.1). So a regular $\mathcal{K}$-analytic space A is normal.

Lemma 3 (Separation Lemma). *In a normal space, and so also in a locally compact Hausdorff space X, for an ordered finite sequence of compact* *sets $\langle K_1, ..., K_n\rangle$ with empty intersection, there is a corresponding ordered finite sequence $\langle U_1, ..., U_n\rangle$ of open sets with empty intersection such that $K_i \subseteq U_i$.*

Proof. Suppose given the compact sequence $\langle K_1, ..., K_n\rangle$. First assume that X is normal. For each i the set K_i is disjoint from the set $K_{-i} := \bigcap_{j\neq i} K_j$. Let $f_i : X \to [0, 1]$ be a continuous function with zero set K_i such that $f_i(K_{-1}) = 1$ and $f := \sum_{i\leq n} f_i$; then $f(x) = 0$ iff $x \in \bigcap_{i\leq n} K_i$, and so $f > 0$ on X. Then $U_i := \{x : f_i(x) < f(x)/n\}$ is open and $K_i \subseteq U_i$. If $x \in \bigcap_{i\leq n} U_i$, then summing the relations $f_i(x) < f(x)/n$ we obtain the contradiction that $f(x) < f(x)$.

Now assume that X is locally compact and Hausdorff; we may choose U open containing $\bigcup_{i\leq n} K_i$ with Y =cl$U$ compact. As Y is normal, we may find V_i in Y separating the K_i as required. Taking $U_i = V_i \cap U$ we obtain the desired separation in X. $\square$

Remark. Here is an alternative proof of the locally-compact case. Suppose otherwise; then there is $\langle K_1, ..., K_n\rangle$ with empty intersection, such that for each corresponding ordered finite sequence $U := \langle U_1, ..., U_n\rangle$ of open sets there is a point x_U in their intersection. Again without loss of generality we may assume the sets K_i all lie in a compact Hausdorff subspace Y. Direct (upwards) the family $\mathcal{U}$ of ordered finite sequences $\langle U_1, ..., U_n\rangle$ of open sets with $K_i \subseteq U_i$ by taking $U \leq V$ iff $U_m \supseteq V_m$ for each m (cf. [Kel] Ch. 2). By compactness, there is a subnet $\langle x_U : U \in \mathcal{U}'\rangle$ with a cluster point x (cf. [Kel] Th. 5.2). We claim that $x \in K_m$ for each m. Fix $i \leq n$; if $x \notin K_i$ then there is an open set V_i with $x \notin$clV_i such that $K_i \subseteq V_i$. For $j \neq i$ put $V_j = Y$. Since $Y\backslash$clV_i is an open nhd (in Y) of x, there is $W \in \mathcal{U}$ such that $x_U \in Y\backslash$clV_i for $U \geq W$ with $U \in \mathcal{U}'$. But for each $U \in \mathcal{U}'$ with $U \geq V$ and $U \geq W$ (i.e. for $U \in \mathcal{U}'$ such that $U_m \subseteq V_m \cap W_m$ for each m), we have $x_U \in \bigcap_m U_m$. In particular, $x_U \in U_i \subseteq V_i$, which contradicts $x_U \notin$clV_i. So $x \in K_m$ for each m, contradicting the non-existence of an open separating sequence. $\square$

We now give the promised generalization of Th.2 from singleton-valued to compact-valued representations.

Theorem 2′ (Multiple $\mathcal{K}$-Analytic Targets - Trace Theorem). *Let X be regular Hausdorff and $A_n = K_n(I)$ be $\mathcal{K}$-analytic in X, with K_n compact-valued and upper-semicontinuous. If F_n is a decreasing sequence of (non-empty) closed sets in X with*

$$F_n \cap \bigcap\nolimits_{m\leq n} K_m(i_1(m), ..., i_n(m)) \neq \emptyset,$$

for some $i(n) = (i_1(n), ...) \in I$ and each n, then $\bigcap_n F_n \cap \bigcap_n K_n(i(n)) \neq \emptyset$.

Proof. If not, write $H_n := K_n(i(n))$. Put $Y = \bigcup_n H_n$. As Y contains the sets $F_n \cap \bigcap_{m\leq n} K_m(I(i_1(m), ..., i_n(m)))$, and regularity is subspace hereditary, we may as well assume that $X = Y$. By compactness, since $\bigcap_n (F_n \cap H_n) = \emptyset$, there is p with $F_p \cap \bigcap_{n\leq p} H_n = \emptyset$. If $F_p \cap H_m = \emptyset$ for some $m \leq p$, then $K_m(i(m)) \subseteq X \backslash F_p$, and so $F_p \cap K_m(I(i(m)|n)) = \emptyset$ for some $n > p+m$. Then $F_n \cap K_m(I(i(m)|n)) = \emptyset$, a contradiction. So the compact set $H'_m := F_p \cap H_m$ is non-empty for each $m \leq p$. Since $\bigcap_{n\leq p} H'_n = \emptyset$, for some open in F_p sets $U_m \supseteq H'_m$ we have $\bigcap_{n\leq p} U_n = \emptyset$ (by Lemma 3 as X is now assumed Lindelöf, and so normal). So for some $n > p$ we have $F_p \cap K_m(i(m)|n) \subseteq U_m$ for each $m \leq p$, and so $F_n \cap \bigcap_{n\leq p} K_m(i(m)|n) = \emptyset$, a contradiction. □

1.3 $\mathcal{K}$-analytic sets: the non-separable variant

Recall that a metric space S is said to be absolutely analytic, or just *analytic,* if it is Souslin-$\mathcal{F}(S^*)$, i.e. is Souslin in its completion S^*. In a non-separable complete metric space X it is not possible to represent a Souslin-$\mathcal{F}(X)$ subset as a $\mathcal{K}$-analytic set relative to $I = \mathbb{N}^\mathbb{N}$. Various aspects of countability need generalization, as we now recall, harking back to A. H. Stone's result that in a metric space, an open cover has a σ-discrete refinement (and to Bing's characterization of metric spaces as regular Hausdorff spaces with σ-discrete bases). We refer to the survey paper [St2] and the more recent [Han-92]; this section is meant as a brief excursion enabling comments to the effect that the primarily separable account here is capable of generalization, although at the cost of an expanded technical apparatus (utilizing the approach to analyticity developed in [HJR]).

An (indexed) family of subsets $\mathcal{H} := \{H_t : t \in T\}$ of a space X is *index-discrete* if every point of X has a neighbourhood meeting at most one set H_t (i.e. meets H_t for at most one index t). The family $\mathcal{H}$ is σ-discrete in X if $\mathcal{H} = \bigcup_n \mathcal{H}_n$ where each indexed family $\mathcal{H}_n = \{H_t^n : t \in T_n\}$ is index-discrete in X.

Denoting by $wt(X)$ the weight of the space X (i.e. the smallest cardinality of a base for the topology), replacing $I = \mathbb{N}^{\mathbb{N}}$ by $J = \kappa^{\mathbb{N}}$ for $\kappa = wt(X)$, consider sets S with the following *extended Souslin representation*, which might be termed more precisely a *σ-discrete κ-Souslin representation*:

$$\bigcup\nolimits_{j\in J} F(j), \text{ where } F(j) := \bigcap\nolimits_{n\in\mathbb{N}} F(j|n),$$

and where the determining system $\langle F(j|n)\rangle$ consist of closed sets with the following two properties:
(i) $\{F(j|n) : j|n \in \kappa^n\}$ is σ-discrete in X for each n,
(ii) $\mathrm{diam}_X F(j|n) < 2^{-n}$, so that $F(j)$ is empty or single-valued, and so compact.
Then, by a theorem of Hansell [Han-73a], the Souslin-$\mathcal{F}(X)$ sets are characterized as those having just such an extended Souslin representation, with $\kappa = wt(X)$. (For other equivalent representations, including a weakening of σ-discreteness of the family in (i) to relative to its union rather than relative to X, see [Han-73b] and [Han-73a].) That is, working relative to J the corresponding extended Souslin sets exhibit properties similar to the $\mathcal{K}$-*analytic* sets *relative* to I. By Hansell's characterization theorem and Nikodym's theorem, again as in §1.1, sets with a σ-discrete κ-Souslin representation have the Baire property. Furthermore, in view of Banach's Localization Principle (§1.1), category considerations may be applied to σ-discrete decompositions of the sets in a family (see definition below), in much the same way as they are applied to countable decompositions in relation to σ-ideals of meagre sets for the purposes of heavy localization (for which see [Ost-LBIII, Th. 1]). Finally, since $F : J \to \mathcal{K}(X)$ above is upper-semicontinuous, the Analytic Cantor Theorems of the preceeding section continue to hold as defined below. Here the mapping F has properties additional to upper-semicontinuity, such

as co-σ-discreteness on J, or index-σ-discreteness on a closed subset of J (see below for these terms).

From this perspective, Hansell, Jayne and Rogers [HJR] (cf. [FH]) develop a theory of $\mathcal{K}$-analytic sets relative to J; a brief summary adequate for the current needs follows. Call a subset A of a Hausdorff space X $\mathcal{K}$-*analytic in* X *relative* to J, if $A = K(J)$ for some upper semicontinuous $K : J \to \mathcal{K}(X)$ such that $\{K(J(j|n)) : j|n \in \kappa^n\}$ is σ-discrete (or, equivalently, σ-discretely decomposable – see below for this term). Note that the σ-discreteness condition is *relative to* X, being a requirement that refers to all points of X, not just those in A. In a Čech-complete space X, such sets are characterized, necessarily *relative to* X, as being 'subparacompact in X' (rather than Lindelöf) and 'Souslin-$\mathcal{F}$ in X'. Here A is 'subparacompact in X ' means that every relatively-open cover of A has a refinement that is σ-discrete in X. (Thus a regular Hausdorff space X is subparacompact in itself iff it is subparacompact in its usual unrelativized sense, for which see [Eng, 5.5.3 (Remark)], or [Bur-1]; cf. [Bur-2].) If X is $\mathcal{K}$-analytic in itself, then its $\mathcal{K}$-analytic subsets are exactly its Souslin-$\mathcal{F}$ sets (see [HJR] Th. 18).

Theorem 1$_J$. *The variants* $1_{\mathbf{C}}$, $1_{\mathbf{Inc}}$,$1_{\mathbf{Tr}}$ *of Theorem 1 above, (i.e. all but* $1_{\mathbf{Cpt}}$*) hold mutatis mutandis as follows:*
(i) *'*$J = \kappa^{\mathbb{N}}$*' replaces '*$I = \mathbb{N}^{\mathbb{N}}$*',*
(ii) *'*$\mathcal{K}$*-analytic relative to* J*' replaces '*$\mathcal{K}$*-analytic relative to* I*'*
(iii) *'complete metric space of weight at most* κ*' replaces 'Polish space'.*

The proofs of Theorem 2 and 2$'$ above used the fact that $\mathcal{K}$-analytic spaces are Lindelöf and that a regular Lindelöf space is normal. But we noted that the separation result in Lemma 3 has an alternative proof when the underlying spaces are regular Hausdorff. We thus have:

Theorem 2$_J$. *For the class of regular Hausdorff spaces the two variants of Theorem 2 hold mutatis mutandis as in Theorem* 1_J.

We close this section by defining some terms referred to above in the discussion.

Definition ([Han-74], §3; cf. [Han-71] §3.1 and [Mich82] Def. 3.3).

1. Call $f : X \to Y$ *base-σ-discrete* (or *co-σ-discrete*) if the image under f of any discrete family in X has a σ-discrete base in Y.

2. Say that an indexed family $\mathcal{A} := \{A_t : t \in T\}$ is *σ-discretely decomposable* if there are index-discrete families $\mathcal{A}_n := \{A_{tn} : t \in T\}$ such that $A_t := \bigcup\{A_{tn} : t \in T\}$.

3. ([Mich82] Def. 3.3.) Call $f : X \to Y$ *index-σ-discrete* if the image under f of any index-discrete family in X is σ-discretely decomposable in Y.

In (3) above f has a stronger property than co-σ-discreteness. (For a proof see [Han-74] Prop. 3.7 (i).) This can at times be more convenient, and in any case the concepts are close, since for metric spaces and κ an infinite cardinal: X is a base-σ-discrete continuous image of $\kappa^{\mathbb{N}}$ iff X is an index-σ-discrete continuous image of a closed subset of $\kappa^{\mathbb{N}}$. (See [Han-92] Th. 4.2, or [Han-74] Th. 4.1.) Moreover, base-σ-discrete continuous maps (in particular index-σ-discrete continuous maps) preserve analyticity ([Han-74] Cor. 4.2); compare also [HJR, Th. 7].

2 Analytically heavy topologies

The theoretical setting of this section is a first topology $\mathcal{T}$ that is Hausdorff and regular, and a second finer topology $\mathcal{T}'$ whose members are 'almost' (i.e. up to a meagre set) generated by the $\mathcal{K}$-analytic sets of $\mathcal{T}$. (See [LMZ] for a monograph treatment of a 'bitopological' view; especially see their Th. 4.2.) However, our motivating examples (listed as Examples A below) are various topologies on $\mathbb{R}$ refining $\mathcal{T}$ the standard (Euclidean) which share the common feature of being generated by analytic sets. These examples are submetrizable: they refine a metrizable topology. In Examples B at the end of this section we consider two examples that are not submetrizable. As noted in §1, analytic sets in $\mathbb{R}$ are all $\mathcal{T}$-circumscribed. The aim below is to see the Gandy-Harrington Theorem as again a van Mill Theorem A, since the former relies on an intersection of analytic sets being non-empty.

We focus here on refinement topologies $\mathcal{T}'$, including the topology $\mathcal{T}$ itself, with the property that for each non-empty $V \in \mathcal{T}'$ there is a $\mathcal{K}$-analytic subset of $(X, \mathcal{T})$, i.e. $A \in \mathcal{A}(\mathcal{T})$ in the notation below, with $\emptyset \neq A \subseteq V$. This natural 'weak base' property is of particular interest, as it falls short of requiring open sets in $\mathcal{T}'$ to be $\mathcal{K}$-analytic in $\mathcal{T}$ (equivalently, by Lemma 1, $\mathcal{F}_\sigma$ in $\mathcal{T}$). A good example, considered below, is the density topology $\mathcal{D}$ of §1. Reflecting the motivation of this paper, we call such refinement topologies *analytically heavy* (see definition below).

In §4 we consider specifically *submetrizable* fine topologies $\mathcal{T}'$, i.e when $\mathcal{T}$ is $\mathcal{T}_d$, the topology generated by a metric d (so that $\mathcal{A}(\mathcal{T}_d)$ comprises the analytic sets). But there we focus on other inter-relations between the two topologies, motivated in Examples C.

Definitions ($\mathcal{K}$-analytically heavy topologies). 1. For $(X, \mathcal{T})$ a topological space denote by $\mathcal{A}(\mathcal{T})$ the family of $\mathcal{K}$-analytic subsets of $(X, \mathcal{T})$.

2. $\mathcal{H}$ is a *topological base* for X if ([Eng] §1.1) $\mathcal{H}$ covers X, and for $H_1, H_2 \in \mathcal{H}$, whenever $x \in H_1 \cap H_2$, there is $H_3 \in \mathcal{H}$ with $x \in H_3 \subseteq H_1 \cap H_2$. We write $\mathcal{G}_\mathcal{H}$ for the topology generated by $\mathcal{H}$.

3. $\mathcal{B}$ is a *weak base* for a topology $\mathcal{T}$ if for each non-empty $V \in \mathcal{T}$ there is $B \in \mathcal{B}$ with $\emptyset \neq B \subseteq V$. In fact, sometimes we need only a *very weak base:* for each non-empty $V \in \mathcal{T}$ there is $B \in \mathcal{B}$ with $\emptyset \neq B \cap V$. See Remark 1 below.

4. Let $(X, \mathcal{T})$ be a regular Hausdorff space and $\mathcal{T}' \supseteq \mathcal{T}$ a refinement topology. We say $\mathcal{T}'$ is *analytically heavy*, or *weakly $\mathcal{K}$-analytically generated* in $\mathcal{T}$, if $\mathcal{T}'$ possesses a weak base $\mathcal{H} \subseteq \mathcal{A}(\mathcal{T})$, all of whose elements have a $\mathcal{T}'$-open representation, i.e. an upper semicontinuous representation $K : I \to \mathcal{K}(X)$ with $K(U) \in \mathcal{T}'$ for U open in I.

Remarks. 1. In this bitopological context we refer to $(X, \mathcal{T})$ as the *ground space* and $(X, \mathcal{T}')$ as the *refinement.*

2. As to $\mathcal{T}'$-open representations, note that for any $A \in \mathcal{G}_d(R)$ that is dense in itself (i.e. is $\mathcal{I}_{Fin}$-heavy, for $\mathcal{I}_{Fin}$ the ideal of finite subsets of X), Kuratowski [Kur-34] (cf. [Jay-Rog] Th. 2.4.1) constructs an upper

semicontinuous representation K with $K(I(i|n))$ a non-empty intersection of an open set $G(i|n)$ and a dense-in-itself closed subset $F(i|n)$ of A of diameter at most 2^{-n}. This construction may be repeated verbatim, ensuring also that $F(i|n)$ has diameter at most 2^{-n} in some complete metric on A and is $\mathcal{I}$-heavy for a larger ideal (i.e. using 'generalized condensation points', cf. [Mar1]). In particular this is available for the weak bases $\mathcal{H}$ corresponding to the $\mathcal{E}l$, $\mathcal{D}$, and $\mathcal{R}$ topologies below, where $\mathcal{H}$ consist of $\mathcal{G}_\delta$ subsets that are $\mathcal{I}$-heavy, for $\mathcal{I} = \mathcal{I}_{Fin}$ and $\mathcal{I} = \mathcal{N}$. (In fact the Kuratowski $\mathcal{F}$-$\mathcal{G}$-representation, being also 'disjoint', yields an $\mathcal{F}_{\sigma\delta}$ representation, cf. [Jay-Rog] Prop. 5.7.3.)

3. If the weak base $\mathcal{H}$ in (2) is actually a base, then we say that $\mathcal{T}$ is a *generalized Gandy-Harrington topology.* (See §2.1 below.)

2.1 Examples A: Submetrizable Examples

1. *A complete separable metric ground space.* For $(X, \mathcal{T}_d)$ with $\mathcal{T}_d$ generated by a complete separable metric d on X, the standard basis $\mathcal{H}$ of all open (analytic) balls yields $\mathcal{G}_\mathcal{H} = \mathcal{T}_d$.

2. (a) *The Gandy-Harrington topology $\mathcal{GH}$.* For $\mathcal{H}$ the countable family of analytic subsets of $\mathbb{R}$ which are *effective* relative to a given real α (i.e. $\Sigma^1_1(\alpha)$), we obtain the Gandy-Harrington topology $\mathcal{GH}$. For background on the standard Gandy-Harrington case $\mathcal{GH}$ and variants, see e.g. [Lou] Prop. 6, or [MK], Section 9.3.

(b) Any subfamily $\mathcal{H}$ of $\mathcal{A}(\mathbb{R})$ closed under intersection, including $\mathcal{A}(\mathbb{R})$ itself, is a base for a topology in the sense of Definition 2 above.

3. *Density topology.* For $\mathcal{I} = \mathcal{N}$, we may take $\mathcal{H} = \mathcal{D} \cap \mathcal{G}_\delta(X)$ as a base for $\mathcal{D}$. (The space is covered, as the usual open intervals are in $\mathcal{D}$. If H_1 and H_2 are $\mathcal{D}$-dense, and so metrically dense on $H_3 = H_1 \cap H_2$ and $\mathcal{N}$-heavy, then H_3 is $\mathcal{N}$-heavy, by Theorem 4 below.) Here $\mathcal{G}_\mathcal{H} = \mathcal{D}$. Unlike in $\mathcal{GH}$, the open sets of $\mathcal{D}$ are not analytic in the ground space, although the basic sets of $\mathcal{H}$ are.

4.(a) *The Ellentuck topology, $\mathcal{E}l$.* The points of this space lie in Cantor space $2^\mathbb{N}$, the latter equipped with the Euclidean topology. The points of

$2^{\mathbb{N}}$ are interpreted as indicator functions of subsets of $\mathbb{N}$. More specifically, one considers only the points corresponding to infinite subsets of $\mathbb{N}$, denoted $[\mathbb{N}]^{\omega}$. This subspace is a $\mathcal{G}_\delta$ in $2^{\mathbb{N}}$, so is topologically complete; indeed, if $\langle f_n \rangle$ enumerates $[\mathbb{N}]^{<\omega}$, the family of all finite subsets of $\mathbb{N}$, then $[\mathbb{N}]^{\omega} = \bigcap_n \{1_S \in 2^{\mathbb{N}} : 1_S \neq 1_{f_n}\}$.

The refinement topology on $[\mathbb{N}]^{\omega}$, called the Ellentuck topology after one of its authors ([Ell], see also Louveau [Lou1] and the more recent [Rear]), is generated by taking for $\mathcal{H}$ the closed subsets $[a, A] := [\mathbb{N}]^{\omega} \cap \{1_S \in 2^{\mathbb{N}} : a \subseteq S \subseteq a \cup A\}$ for a finite and $A \subseteq \mathbb{N} \backslash \{0, 1, ..., \max a\}$ infinite. Note that $A = \mathbb{N} \backslash \{0, 1, ..., \max a\}$ gives a set in the usual Cantor basis.

If $\mathcal{I} = \{\emptyset\}$, then (not ulike the case 2(b) above) $\mathcal{H}$ is $\mathcal{I}$-heavy. The space is Choquet and so Baire ([Kech], 19.13 and 8.12); the latter will be confirmed in Theorem 3 below. The topology yields a 'short-cut' for a proof of the Silver-Mathias Theorem that analytic sets (in the ground space) have the Ramsey property – see the remarks below.

(b) Unlike $\mathcal{GH}$, the Ellentuck topology is generated by a continuum of analytic (in fact $\mathcal{G}_\delta$) sets; a countable *effective* coarsening of significance has been studied in [Avi].

5. *O'Malley's r-topology* (or *resolvable*-topology). To study approximate differentiability of real-valued functions, O'Malley [O] introduces the r-topology $\mathcal{R}$ on $\mathbb{R}$ with $\mathcal{R} \subseteq \mathcal{D}$; it is generated by taking as base $\mathcal{B} := \mathcal{D} \cap \mathcal{G}_\delta \cap \mathcal{F}_\sigma$ the sets of $\mathcal{D}$ that are ambiguously both $\mathcal{G}_\delta$ and $\mathcal{F}_\sigma$ in the real line. (For these, see also [St1] Th. 10. Recall that in a complete metric space a set that is both $\mathcal{G}_\delta$ and $\mathcal{F}_\sigma$ may be characterized as resolvable – see [Kur-1] §12. III, V.) For other aspects see §4. It is a generalized Gandy-Harrington topology, avant la lettre.

2.2 Generalized Gandy-Harrington Theorem

The argument for Theorem 3 below repeatedly uses the fact that if $\bigcup_n A_n \cap B \neq \emptyset$, then $A_n \cap B \neq \emptyset$ for some n. We view this as saying that $\mathcal{I} = \{\emptyset\}$ has the localization property and $\bigcup_n A_n$ is $\mathcal{I}$-heavy on B. This motivates an $\mathcal{I}$-refinement variant for a general σ-ideal, which we give as Theorem $3'$ in

§4.

Theorem 3 (Generalized Gandy-Harrington Theorem). *In a regular Hausdorff space, if $\mathcal{T}'$ is an analytically heavy refinement topology of $\mathcal{T}$ (i.e. possessing a weak base $\mathcal{H} \subseteq \mathcal{A}(\mathcal{T}) \cap \mathcal{T}'$ whose elements have $\mathcal{T}'$-open representation), then $\mathcal{T}'$ is Baire.*

In particular, this applies to a Polish space, the Gandy-Harrington $\mathcal{GH}$, the density $\mathcal{D}$, the Ellentuck $\mathcal{E}l$ and the O'Malley $\mathcal{R}$ topologies.

Proof. We use the notation of §1.1 and also put $I_n := \mathbb{N}^n = \{i|n : i \in I\}$. For each n, let W_n be dense and open in $\mathcal{T}'$. Suppose inductively that for all $m \leq n$ there are upper-semicontinuous compact-valued maps $G_m : I \to X$ which are $\mathcal{T}'$-open such that $G_m(I) \subseteq W_m$ with $G_m(I) \in \mathcal{H}$, $\sigma_n(m) \in I_n$ for $m \leq n$, and

$$G_1(\sigma_n(1)) \cap ... \cap G_n(\sigma_n(n)) \neq \emptyset.$$

Then

$$U_n := G_1(\sigma_n(1)) \cap ... \cap G_n(\sigma_n(n)) \neq \emptyset \text{ and } U_n \in \mathcal{T}'.$$

As U_n is non-empty and open in $\mathcal{T}'$ and W_{n+1} is $\mathcal{T}'$-dense, $W_{n+1} \cap U_n \neq \emptyset$. Since $\mathcal{H}$ is a weak base, there is $A_{n+1} \in \mathcal{H}$ with $\emptyset \neq A_{n+1} \subseteq (W_{n+1} \cap U_n) \subseteq W_{n+1}$ and in particular $A_{n+1} \cap U_n \neq \emptyset$. Taking $A_{n+1} = G_{n+1}(I)$ with G_{n+1} a $\mathcal{T}'$-open representation and noting that $G_{n+1}(I) = \bigcup\{G_{n+1}(\sigma) : \sigma \in I_n\}$, there is $\sigma_n(n+1) \in I_n$ such that

$$G_1(\sigma_n(1)) \cap ... \cap G_n(\sigma_n(n)) \cap G_{n+1}(\sigma_n(n+1)) \neq \emptyset.$$

But $G_m(\sigma_n(m)) = \bigcup_k G_m(\sigma_n(m), k)$. So there are extensions $\sigma_{n+1}(m)$ of $\sigma_n(m)$ for each $m \leq n+1$ such that

$$G_1(\sigma_{n+1}(1)) \cap ... \cap G_{n+1}(\sigma_{n+1}(n)) \neq \emptyset.$$

This verifies the induction step. So for each m there is $i(m) \in I$ with $i(m)|n = \sigma_n(m)$ for each n. Applying Theorem 2′ in the ground space (taking $F_n = X$), we have

$$\emptyset \neq \bigcap_m G_m(i(m)) \subseteq \bigcap_m A_m \subseteq \bigcap_m W_m.$$

For W an arbitrary non-empty open set in $\mathcal{T}'$, as the set $W_n \cap W$ is $\mathcal{T}'$-dense on W, we conclude by the preceeding argument that $\emptyset \neq W \cap \bigcap_m W_m$. So $\mathcal{T}'$ is Baire. $\square$

Remarks. 1. In the case of the Gandy-Harrington topology, the members of $\mathcal{H}$ are analytic sets with representations K_m such that each of the sets $K_m(i|n)$ is also in $\mathcal{H}$, so open by fiat in $\mathcal{T}' = \mathcal{GH}$. That is, the representations are $\mathcal{T}'$-open.

2. In the proof above, when $\mathcal{T}' = \mathcal{T}$ (with $\mathcal{T}$ regular) and $\mathcal{H}$ is a *very weak base*, then one may first select $\emptyset \neq V_{n+1} \in \mathcal{T}$ with $V_{n+1} \subseteq \mathrm{cl}V_{n+1} \subseteq W_{n+1} \cap U_n$; so if $A_{n+1} \cap (W_{n+1} \cap V_n) \neq \emptyset$ for some $A_{n+1} \in \mathcal{H}$, then $\emptyset \neq A'_{n+1} := A_{n+1} \cap \mathrm{cl}V_{n+1} \subseteq W_{n+1}$ and A'_{n+1} is analytic. The proof may be modified provided both A'_n and A_n are $\mathcal{T}'$-circumscribed.

3. There is a natural connection between topologies and the partially ordered sets in forcing (see [Je]). Category and measure correspond respectively to generic constructions of *Cohen reals* and *Solovay reals* (or random reals); in like manner the Gandy-Harrington topology corresponds to generic constructions (of minimal 'degrees') via Gandy forcing – see e.g. [Mil] Ch. 30 for a modern treatment.

4. The Ellentuck topology corresponds to the generic construction of *Mathias reals* (i.e. via Mathias forcing, [Math]), in relation to a Ramsey property of a class of subsets $\mathcal{S}$ of reals. The fine topology provides a direct interpretation of a combinatorial property: $\mathcal{S}$ has the Ramsey property iff $\mathcal{S}$ has the Baire property under the Ellentuck topology. In particular, as sets closed in the ground space are closed in $\mathcal{E}l$, it follows from the Nikodym theorem (see §1.1) that analytic sets in the ground space have the Baire property in $\mathcal{E}l$, and hence also the Ramsey property, a fact originally proved by Silver, and generalized by Mathias, but these authors employ techniques from mathematical logic. (See [Ell] and [Lou1] for the topological approach.)

To define the combinatorial property, identify the real 1_S with S and $[a, A]$ with $\{S : a \subseteq S \subseteq a \cup A\}$. A set $\mathcal{S}$ of 'reals' S in $[\mathbb{N}]^\omega$ has the *complete Ramsey property* if, for any finite a and infinite A (as previously), there is an infinite $A' \subseteq A$ such that either $[a, A'] \subseteq \mathcal{S}$ or $[a, A'] \subseteq [\mathbb{N}]^\omega \backslash \mathcal{S}$. In the language of colours, $\mathcal{S}$ paints the infinite subsets of $\mathbb{N}$ with two colours (according to inclusion or exclusion), and exhibits the following monochromatic property for each finite a : any infinite A avoiding $\{0, ..., \max a\}$ has a monochromatic subset.

See [Paw] for a topological proof of a parametrized version.

5. See [Kech] Th. 25.18 and 25.19 for Becker's Theorem ([Bec] Th. 1.2), that for a second-countable extension $\mathcal{T}$ of a Polish space $(X, \mathcal{T}_d)$, the topology $\mathcal{T}$ is strong Choquet iff $\mathcal{T} \subseteq \mathcal{A}(\mathcal{T}_d)$. We review this remark in another context in the Remark 5 in §5.1.

2.3 Examples B: Two non-submetrizable examples.

A regular Hausdorff space which is locally countable and locally compact is (being locally metrizable and and locally compact) locally a completely metrizable space, so Baire. This illustrates the theorem, as such a space is analytically heavy, because it possesses a weak base of the kind considered above. Indeed, in a locally metric space a one-point isolated set has $\mathcal{G}$-open analytic representation, from which it follows that a countable light-part has Kuratowski $\mathcal{F}$-$\mathcal{G}$-representation that is $\mathcal{G}$-open.

1. In the case of the set of countable ordinals, ω_1 with the interval (order) topology $\mathcal{T}_<$, every dense open set contains the (isolated) set of successor ordinals, so the space is Baire rather obviously. We will refer to the fact that under $\mathcal{T}_<$ any two uncountable sets have a common limit point (so, if closed, must meet).

2. We pass to a refinement (so also a locally countable topology), that is locally compact and separable: the *Ostaszewski space* $(\omega_1, \mathcal{T}_\clubsuit)$, constructed under the assumption of an additional set-theoretic axiom (for which see [Ost-1], or [JR] §5.3, [MER]). Here every open set (and so every closed set) is either countable or co-countable. So any two uncountable closed sets, being both co-countable, must meet. The space is Baire, being locally completely metrizable, as above.

Neither example is submetrizable. For if either topology were to refine a metrizable topology $\mathcal{T}_d$, then $\mathcal{T}_d$ would need to be separable. For $\mathcal{T}_\clubsuit$, which is separable, this is immediate. For $\mathcal{T}_<$, if $\mathcal{T}_d$ were non-separable, there would exist an uncountable ε-separated subset Z, for some $\varepsilon > 0$. The $\mathcal{T}_d$-closures of any two uncountable disjoint subsets Z_1, Z_2 of Z, being $\mathcal{T}_<$-closed and uncountable would, as observed earlier, have a point in common, x say, contradicting ε-separation (since $B_{\varepsilon/2}(x)$ then contains points $z_i \in Z_i$ that are not ε-separated). In the separable topology $\mathcal{T}_d$ the set $H(\omega_1)$ of condensation points (i.e. the $\mathcal{I}$-heavy points, for $\mathcal{I}$ the σ-ideal of countable sets) forms a non-empty perfect set, and so $(\omega_1, \mathcal{T}_d)$ contains two disjoint uncountable closed sets. But these two $\mathcal{T}_d$-closed sets are also closed in the refinement topologies $\mathcal{T}_\clubsuit$ or $\mathcal{T}_<$, and so must meet, contradicting disjointness.

We consider in Examples C of §4 the *Kunen line*, a related submetrizable example which is also locally compact and locally countable (a refinement topology on $\mathbb{R}$). Evidently the theorem applies.

3 Analytic Baire from Analytic Cantor

We recall some results of van Mill from [vM]. The first is adapted to a general σ-ideal $\mathcal{I}$, but, except in §4, we usually take $\mathcal{I}$ to be $\mathcal{M}$.

Lemma 4 ([vM], Prop. 1.1)**.** *For A dense and $\mathcal{I}$-heavy and $M \in \mathcal{I}$, the set $A \backslash M$ is dense and $\mathcal{I}$-heavy.*

Proof. If not, then for some V that is non-empty open $(A \backslash M) \cap V \in \mathcal{I}$, which includes the possibility that $(A \backslash M) \cap V = \emptyset$. But then $V \cap A \subseteq M \cup [(A \backslash M) \cap V] \in \mathcal{I}$, a contradiction to A being $\mathcal{I}$-heavy. $\square$

The following result for A analytic goes back to Levi [Levi] – see §5.4; here we follow the form which van Mill [vM] uses, but must disaggregate the result into two components (Propositions L1 and L2), corresponding to the domain and image of an upper-semicontinuous representation of a heavy and dense analytic set. The Corollary below verifies that their recombination implies van Mill's original result. Let $\mathcal{I}$ be a σ-ideal; except in §4, this will usually be $\mathcal{M}$. The next definition is motivated by Levi's extraction (sketched in

Comment 5.4) from a given continuous map K of a 'submap' $K|S$ with the direct Baire property (taking open sets to Baire sets), whence the term $\mathcal{I}$-*faithful* below. However, Proposition L1 identifies a *minimal* closed domain S defining the submap; thus the submap is irreducible over G in the sense that, for any proper closed subset S' of S, the image $K(S')$ is not equal modulo $\mathcal{I}$ to $K(S)$ (as $G \cap K(S \backslash S') \notin \mathcal{I}$); cf. [Eng] Ex. 3.1.C. One may equally well term the submap $\mathcal{I}$-*irreducible* on G (as in [Ost-ACE]).

Definition. For a $\mathcal{K}$-analytic set A in X with upper-semicontinuous representation $K : I \to A$ and $A \cap G \notin \mathcal{I}$ for some open $G \subseteq X$, say that K is $\mathcal{I}$-*faithful on G over S* (or $\mathcal{I}$-*irreducible* on G) if $G \cap K(S \cap V) \notin \mathcal{I}$ for each open V meeting S.

Proposition L1 ($\mathcal{I}$-faithful/$\mathcal{I}$-irreducible representation). *For $A = K(I)$, with $K : I \to \wp(X)$ arbitrary, and $G \subseteq X$ such that $A \cap G \notin \mathcal{I}$, there is a minimal closed subspace S such that $G \cap K(S \cap V) \notin \mathcal{I}$ for each open V meeting S. In particular, $A' := K(S)$ differs from A on G by a set in $\mathcal{I}$. Furthermore, if A is dense and $\mathcal{I}$-heavy in G, then so is A'.*

Proof. For $A = K(I)$ as above, let $W := \bigcup\{V : V$ is open and $K(V) \cap G \in \mathcal{I}\}$. If $\mathcal{B}$ is a countable basis for I, then

$$W := \bigcup\{V : V \in \mathcal{B} \text{ is open and } K(V) \cap G \in \mathcal{I}\},$$

and so

$$G \cap K(W) := \bigcup\{G \cap K(V) : V \in \mathcal{B} \text{ is open and } K(V) \cap G \in \mathcal{I}\}$$

is in $\mathcal{I}$. Note that W is identifiable as the maximal open subset of I such that if $K(V) \cap G \in \mathcal{I}$ for any open $V \in \mathcal{B}$ then $V \subseteq W$.

Now $S = I \backslash W$ is topologically complete. For $V \subseteq I \backslash W$ non-empty and open in $I \backslash W$, write $V = (I \backslash W) \cap U$ with U open in I. As $U \subseteq V \cup W$, for $K(V) \cap G \in \mathcal{I}$ one has $G \cap K(U) \subseteq G \cap (K(V) \cup K(W)) \in \mathcal{I}$. So $G \cap K(U) \in \mathcal{I}$, and so $U \subseteq W$ and $V = \emptyset$. So $K(V) \cap G \notin \mathcal{I}$. Thus S is the minimal closed subset of I such that $G \cap K(V) \notin \mathcal{I}$ for any open V meeting S. Finally, $A' := K(S) \notin \mathcal{I}$ and $G \cap (A \backslash A') \subseteq G \cap K(W) \in \mathcal{I}$.

The last assertion follows from Lemma 4 applied in the space G, since $G \cap (A \backslash A') \in \mathcal{I}$. $\square$

The following result rests on the Category Localization Lemma of §1.

Proposition L2 (Heavy Localization). *If A is non-meagre in X, then there is a non-empty open set H, namely* $\mathrm{int}(\mathrm{cl}(A))$, *such that A is heavy and dense on H, i.e. $A \cap H$ is heavy and dense in H.*

Proof. Indeed $H :=\mathrm{int}(\mathrm{cl}(A))$ is non-empty, as A is not nowhere dense. Also $A \cap H$ is dense in H. Put

$$M_H(A) := \bigcup \{G \cap A : G \subseteq H, G \text{ open and } G \cap A \text{ meagre}\},$$

which is meagre. So $B := (A \cap H) \backslash M_H(A)$ is heavy on H. Indeed, if for G non-empty and open in H the set $B \cap G$ is non-empty and meagre, then, since $B \cap G = (A \cap G) \backslash M_H(A)$, one has $A \cap G \subseteq (B \cap G) \cup M_H(A)$, which is meagre; but then $A \cap G \subseteq M_H(A)$, so $B \cap G = \emptyset$, a contradiction. A fortiori, the larger set $A \cap G$ is non-empty and non-meagre, and so $A \cap H$ is heavy on H. $\square$

Corollary (Faithful/Irreducible dense-heavy representation). *In the setting of Propositions L1 and L2 with $G = H = X$ and $\mathcal{I} = \mathcal{M}$, if $K(I)$ is dense and $\mathcal{M}$-heavy, then there is a minimal closed subspace S of such that $K(S \cap V)$ is non-meagre for each open V meeting S.*

Proof. As $A' = K(S)$ differs from A on X by a meagre set, $H \cap A'$ is dense and heavy, by Lemma 4. $\square$

For completeness, we note the following result, which is an immediate corollary (a generalization of S. Levi Th. 6 in [Levi]; cf. §5.4; see also [Ost-AB] for further generalizations). It motivates work in §4, where arguments are conducted modulo meagre sets to bring completeness into focus.

The notion of 'almost completeness' is due to Frolík in [Frol-60] (but its name to Michael [Mich91] – see also [AL] and [BOst-N]). See [Wh] Th. 11 for its relevance to the Choquet game below. See also the remarks on Solecki's dichotomy at the end of §3.2.

Proposition L3 (Almost-Completeness Theorem). *A non-meagre $\mathcal{K}$-analytic space A contains a dense $\mathcal{G}_\delta$ which is, for A completely regular, Čech-complete.*

Proof. In view of the second assertion we begin by working in any space X in which A is dense. As $\mathrm{cl}A\backslash\mathrm{intcl}A$ is nowhere dense, so is $A\backslash\mathrm{intcl}(A)$; by Proposition L2, $A\cap\mathrm{intcl}(A)$ is dense heavy in $\mathrm{intcl}(A)$, so contains $G\backslash M$ for some dense open subset G and meagre set M, as in Lemma 2 (by the Baire property implied by Nikodym's Theorem, since a $\mathcal{K}$-analytic set is Sousin-$\mathcal{F}(X)$ – see the remarks in §1.1 – and since the intersection of two Baire sets is Baire). Since $M\cup(A\backslash\mathrm{intcl}(A))$ is contained in a meagre $\mathcal{F}_\sigma$, say in $\bigcup_n F_n$ with each F_n nowhere dense and closed, we have $A\supseteq H:=\bigcap_n(G\backslash F_n)$, which is a dense $\mathcal{G}_\delta$.

For A completely regular, we may conduct the argument in $X=\beta A$, the Stone-Čech compactification (in which A is dense). Then H is a dense $\mathcal{G}_\delta$ in βA, so H is Čech-complete. □

The existence of a dense completely metrizable subspace in a classically analytic space is a result that implicitly goes back to Kuratowski (by [Kur-1] IV.2 p. 88, because a classically analytic set is Baire in the restricted sense – Cor. 1 p. 482). Since an analytic space is a continuous image, the result may be viewed as an 'almost preservation' result for complete metrizability under continuity, in the spirit of the classical theorem of Hausdorff (resp. Vaĭnšteĭn) on the preservation of complete metrizability by open (resp. closed) mappings – see [HP] and [H] for the most recent improvements (based on [Mich86, §6]) and the literature. We note that Michael [Mich91, Prop. 6.5] shows that almost completeness is preserved by demi-open maps (i.e. continuous maps under which inverse images of dense open sets are dense).

3.1 Choquet games: Oxtoby and van Mill Theorems

We now embark on an improvement of Lemma 2 in Theorem 4 below. The proof uses heaviness to obtain a non-meagre part of A in some region G_0 and follows this up using Propositions L1 and L2 repeatedly in smaller regions. The argument has a game-theoretic format. To bring this out we need a convenient definition.

Definitions (Choquet game $\mathrm{Ch}(X;A)$ with target A). 1. For X a topological space and $A \subseteq X$, the (weak) *Choquet game* $\mathrm{Ch}(X;A)$ has two players, the first to move called β and the second called α, who alternately select non-empty open sets U_n in a topological space X such that their moves create a descending sequence $U_0 \supseteq U_1 \supseteq U_2 \supseteq ...$ as their play, and gives a *win* to player α iff $A \cap \bigcap_n U_n \neq \emptyset$.

In a metric context, one of many equivalent definitions of the game requires that $\emptyset \neq \bigcap_n U_n \subseteq A$. (Dropping the non-emptiness, one obtains the *Banach-Mazur game;* for historical background see [Tel].)

2. (i) Player α has a *winning strategy* if there is a function $\tau(U_0, ..., U_{2n-1}, U_{2n})$ defined on even-lengthed descending sequences of open sets U_i of X yielding an open set $\tau(U_0, ..., U_{2n}) \subseteq U_{2n}$ such that the play arising from the choice $U_{2n+1} = \tau(U_0, ..., U_{2n})$ gives a win to α no matter how β plays. (That is, $A \cap \bigcap_n U_n \neq \emptyset$ for all $\{U_{2i}\}_{i\in\omega}$.)

(ii) Player α has a *stationary winning strategy* if there is a function $\tau(U)$ defined on the open sets U of X yielding an open set $\tau(U) \subseteq U$ such that the play arising from the choices $U_{2n+1} = \tau(U_{2n})$ gives a win to α no matter how β plays.

3. Say that A is a *weakly α-favourable* (resp. *stationarily weakly α-favourable*) subspace if player α has a winning strategy (stationary winning strategy) in the game above.

Remarks. 1. We note *Oxtoby's Theorem* that α has a winning strategy in the weak Choquet game $\mathrm{Ch}(\mathbb{R};A)$ iff A is co-meagre (see [Oxt2] Ch. 6; [Kech] 21.C).

2. The following *strong Choquet* game strengthens the game Ch(X, A): player β selects not only U_{2n} but also additionally a point x_{2n}, and α's response U_{2n+1} must satisfy $x_{2n} \in U_{2n+1} \subseteq U_{2n}$, yielding a play $U = \langle U_n \rangle$. Thus β can choose whether or not α may play in the heavy part of U_{2n}. The winning rule here for player α is that $A \cap \bigcap_n U_n \neq \emptyset$. The topology on A is said to be strongly Choquet if α has a winning strategy, i.e. ensures that the open sets played have a non-empty intersection. See e.g. [Kech] §8D, which includes the result that a second-countable topology on X is Polish iff the topology is T_1, regular and strongly Choquet. (Note also Th. 8.33 there, relating β-favourability to a non-empty light part of A when the topology on X is submetrizable – refines a metrizable topology.)

3. The letter β aptly calls to mind that the first player sets the 'Baire category-test' on the set A, which α attempts to pass. In the weak game β cannot prevent α from playing into a dense and heavy part of U_{2n} and this feature is exploited in the proofs of Theorems 4 and 5 in the construction of a winning strategy for α.

4. Theorems 4 and 5 below establish that weak α-favourability is implied by certain forms of analyticity. See §5.1 for other Choquet games which enable α-favourability to be an ingredient of a necessary and sufficient condition characterizing certain topological spaces. (These generalize Choquet's result [Choq] that among metrizable spaces weak α-favourability characterizes almost completeness – cf. Proposition L3, and Remark 1 above.) Indeed our results may be interpreted as verifying that certain other Choquet games yield winning strategies for α.

Lemma 5 (cf. [Wh] Th. (1); [HM], [Kech] 8.H, 21C). *If X is weakly α-favourable, then X is a Baire space.*

Proof. Here $A = X$. For $\{U_n\}_{n\in\omega}$ dense open in X, let W be any non-empty open set, and let β play $G_0 := U_0 \cap W$, which is non-empty. Player α responds under the winning strategy τ with T_1 such that $T_1 \subseteq W$. As $U_0 \cap U_1$ is dense open, the set $G_1 := T_1 \cap U_0 \cap U_1$ is open and non-empty. We let β respond to T_1 with G_1. Inductively, α plays T_n under τ and β responds with

$G_n := T_n \cap U_0 \cap U_1 \cap ...U_n \subseteq T_n$. Since τ is a winning strategy, $\bigcap_n T_n$ contains a point in $\bigcap_n G_n \subseteq W \cap \bigcap_n U_n$. So X is Baire. $\square$

Theorem 4 (Generalized Oxtoby Theorem: α-favourability). *For a regular space X, if A is $\mathcal{K}$-analytic, heavy and dense in X, then X and A are weakly α-favourable spaces and so Baire spaces. In particular, this holds for A $\mathcal{D}$-dense and $\mathcal{N}$-heavy.*

Proof. Suppose that $A = K(I)$ is dense and heavy, with K a fixed upper-semicontinuous representation of A. In the proof below we need to describe explicitly a strategy τ for player α. We let player β select sets G_n, and player α will respond with sets $T_n := \tau(G_n)$. (Here we suppress earlier data from the play.) So we need to fix a countable basis $\mathcal{B}$ of clopen sets for I and an enumeration $\{B_m\}$ of $\mathcal{B}$. We also fix a (possibly) transfinite enumeration of a base in X.

To apply Theorem 1 we will use Propositions L1 and L2 inductively.

Let G_0 any open non-empty subset of X. We select closed subspaces S_n in I, open subsets V_n of $\mathcal{B}$ meeting S_n with diam $V_n < 2^{-n}$, and open sets G_n in X with cl$G_{n+1} \subseteq G_n$. Here S_n is chosen minimal in S_{n-1} with respect to $K(S_n \cap V)$ being dense and heavy on $G_n \cap U_1 ... \cap U_n$ for each $V \in \mathcal{B}$ meeting S_n, in particular $K(S_n \cap V_n)$ being dense and heavy on $G_n \cap U_1 ... \cap U_n$. Since the sets $S_n \cap V_n$ are closed, their intersection contains a single point i in I. So Theorem 1_{C} applies, and $K(i) \cap \bigcap_n G_n \neq \emptyset$ contains a point of $\bigcap_n U_n$.

Given G_0 played by β, we note that as A is dense and heavy, $A \cap G_0$ is non-meagre. We begin the induction using Proposition L1 to take S_1 minimal closed in I with $K(V \cap S_1)$ non-meagre on G_0 for every $V \in \mathcal{B}$ meeting S_1. Referring to the enumeration of $\mathcal{B}$, we pick the first element $V_1 \in \mathcal{B}$ with diam $< 2^{-1}$ meeting S_1. Since $K(V_1 \cap S_1)$ is not meagre on G_0, it is not nowhere dense on G_0. So $T_1 := G_0 \cap$int(cl $K(V_1 \cap S_1)$) is non-empty, and $K(V_1 \cap S_1)$ is dense and heavy on T_1, by Proposition L2. We define $\tau(G_0)$ to be T_1.

Now β responds with $G_1 \subseteq T_1$. By regularity, pick the first non-empty $\tilde{G}_1 \subseteq$cl$\tilde{G}_1 \subseteq G_1 \subseteq T_1$ in the fixed base of X. As $K(V_1 \cap S_1)$ is heavy and dense on T_1, it is non-meagre on T_1, so again by Proposition L1 there is a minimal closed subset S_2 of S_1 such that $K(V \cap S_2)$ is non-meagre on $\tilde{G}_1$ for each $V \in \mathcal{B}$ meeting S_2. The process may be continued, as we now verify.

Inductively, suppose given the play by β of $G_n \subseteq T_n$ and a canonical choice (relative to the base of X) of $\tilde{G}_n \subseteq \mathrm{cl}\tilde{G}_n \subseteq G_n \subseteq T_n$. As $K(V_n \cap S_n)$ is heavy and dense on T_n, it is non-meagre on $\tilde{G}_n$, so again by Proposition L1 there is a minimal closed subset S_{n+1} of S_n such that $K(V \cap S_{n+1})$ is non-meagre on $\tilde{G}_n$ for each $V \in \mathcal{B}$ meeting S_{n+1}. Choose the first V_{n+1} in $\mathcal{B}$ with diam $< 2^{-n-1}$ meeting S_{n+1}, and note that $T_{n+1} := \tilde{G}_n \cap \mathrm{int}(\mathrm{cl}\ K(V_{n+1} \cap S_{n+1}))$ is a non-empty subset with $\mathrm{cl}T_{n+1} \subseteq G_n$. We define $\tau(G_n)$ to be T_{n+1} and note that $K(V_{n+1} \cap S_{n+1})$ is dense and heavy on T_{n+1}. This completes the inductive step.

By Lemma 5 the space is Baire. The measure analogue follows by Lemma $2'$. $\square$

The argument relies on knowing the irreducible representation, so does not yield stationarily weak α-favourability. However, one may recapture stationarity by passing to an 'unfolding' of the game – cf. [Kech] §21.D.

We now prove a stronger form of the Analytic Baire Theorem for $\mathcal{K}$-analytic sets using Theorem 3. Heaviness is again used to start a descent into parts of a $\mathcal{K}$-analytic set and Propositions L1 and L2 continue the process. Again the argument is game-theoretic, for which we need a further definition.

Definition (Multiple-target Choquet game $\mathrm{Ch}(X; \{A_m : m \in \kappa\})$**).** Denote by ω the least infinite ordinal. For $\kappa \leq \omega$, X a topological space, and sets $A_m \subseteq X$ defined for $m \in \kappa$, the *Choquet game* $\mathrm{Ch}(X; \{A_m : m \in \kappa\})$ has two players, the first to move called β and the second called α, who alternately select non-empty open sets U_n in a topological space X such that their moves create a descending sequence $U_0 \supseteq U_1 \supseteq U_2 \supseteq \ldots$ as their play, and give a *win* to player α iff $\bigcap_{n\in\omega} U_n \cap \bigcap_{m\in\kappa} A_m \neq \emptyset$.

Winning strategies and stationary winning strategies are defined as before.

Note that in the metric context, by insisting on shrinking diameters, one may interpret this as a game on X^κ, with target the product set $\hat{A} := \prod_{m\in\kappa} A_m$, the moves U_n coding for corresponding product sets, with $\hat{U}_n := (U_n)^\kappa$ for κ finite, or $\hat{U}_n := (U_n)^n \times X^{\omega\backslash n}$ for κ infinite, since $\bigcap_n(\hat{U}_n \cap \hat{A}) = \prod_{m\in\kappa}(\bigcap_{n\in\omega}(U_n \cap A_m))$. We can no longer draw this conclusion given the possibly distinct points $u_m \in A_m \cap \bigcap_{n\in\omega} U_n$. This motivates the stronger property in the next definition, facilitating an improvement on Theorem 4, and explains why the improvement is not an equivalence.

Say that the κ-product space $\hat{A}$ is a *weakly α-favourable* (resp. *weakly stationarily α-favourable*) subspace if player α has a winning strategy (stationary winning strategy) in the multiple-target game above.

Theorem 5 (**Generalized Oxtoby-van Mill Theorem**, cf. van Mill [vM], Prop. 2.2). *For X regular and Hausdorff and A_n dense, $\mathcal{M}$-heavy, and $\mathcal{K}$-analytic in X, the product space $\hat{A} := \prod_m A_m$ is weakly α-favourable; in particular $\bigcap_n A_n$ is dense and $\mathcal{M}$-heavy. In particular, this holds for A_m $\mathcal{D}$-dense and $\mathcal{N}$-heavy.*

Proof (from the Analytic Cantor Theorem 2). As each set A_n is dense in X, each is dense in their union, so we may suppose that $X = \bigcup A_n$ and so that X is in fact normal (since a regular Lindelöf space is normal). This means that we will be able to apply Theorem 2.

We consider first the case $\kappa = 2$ and determine a winning strategy π for α when $A_1 = K_1(I)$ and $A_2 = K_2(I)$ are heavy and dense. Then $A_1 \cap A_2$ is heavy and dense. One then proceeds for larger κ in a similar way.

As before, let $\mathcal{B}$ be an enumerated basis in I, and fix a (possibly transfinitely) enumerated basis for X.

Let U_0 be an arbitrary non-empty open set played by β.

As A_1 is dense and heavy, A_1 is dense and heavy on U_0, and so by Proposition L1 there exists a minimal closed S_1 in I with $K_1(S_1 \cap V)$ non-meagre on U_0 for every $V \in \mathcal{B}$ meeting S_1.

We pick the first $V_1 \in \mathcal{B}$ with diameter $< 2^{-1}$ meeting S_1. Now $K_1(S_1 \cap V_1)$ is non-meagre in U_0. Put $T_1 := U_0 \cap \mathrm{int}(\mathrm{cl} K_1(S_1 \cap V_1))$; then $K_1(S_1 \cap V_1)$ is heavy and dense on T_1. By regularity, we may pick a first non-empty G_0'

with $G_0' \subseteq \mathrm{cl} G_0' \subseteq T_1 \subseteq U_0$. Then $K_1(S_1 \cap V_1)$ is heavy and dense in G_0'.

As A_2 is heavy and dense, A_2 is dense and heavy on G_0', and so there is a minimal closed S_1' in I such that $K_2(S_1' \cap V)$ is non-meagre in G_0' for every $V \in \mathcal{B}$ meeting S_1'. Pick the first $V_1' \in \mathcal{B}$ with diam $< 2^{-1}$ such that $K_2(S_1' \cap V_1')$ is non-meagre in G_0'. Put $T_1' = G_0' \cap \mathrm{intcl} K_2(S_1' \cap V_1')$, which is non-empty; then $K_2(S_1' \cap V_1')$ is heavy and dense on T_1'. By regularity, we may pick the first (in the basis for X) non-empty G_1 with $G_1 \subseteq \mathrm{cl} G_1 \subseteq T_1' \subseteq G_0'$. We let $\pi(U_0)$ be G_1.

We continue with this two-sets-at-a-time process, inductively selecting, in response to a given sequence of sets U_n, decreasing 'minimal' closed subsets S_n, S_n' and 'earliest' open sets V_n, V_n' in $\mathcal{B}$ with diameters less than 2^{-n} and in X, open sets T_n, T_n' and $G_n \supseteq G_n'$ such that (i)-(iv) hold below.

(These sets correspond to α playing G_i (after computing G_i') and player β's response to that move of $U_i \subseteq G_i$.)

(i) $G_n \subseteq \mathrm{cl} G_n \subseteq T_n' \subseteq G_n'$ and $G_n' \subseteq \mathrm{cl} G_n' \subseteq T_n \subseteq G_{n-1}$;

(ii) $T_n := U_{n-1} \cap \mathrm{int}(\mathrm{cl} K_1(S_n \cap V_n))$ and $T_n' := G_{n-1}' \cap \mathrm{int}(\mathrm{cl} K_1(S_n' \cap V_n'))$;

(iii) $K_1(S_n \cap V_n)$ is heavy and dense on T_n;

(iv) $K_2(S_n' \cap V_n')$ is heavy and dense on T_n'.

We verify the inductive step from n odd.

Suppose that β plays $U = U_n \subseteq G_n$.

By the two clauses of (i), $G_n \subseteq T_n$, so $U \subseteq T_n$ and so by (iii), $K_1(S_n \cap V_n)$ is non-meagre on U, so by Proposition L1 there is a minimal closed set S_{n+1} in S_n such that $K_1(S_{n+1} \cap V)$ is non-meagre on U for every $V \in \mathcal{B}$ meeting S_{n+1}. So there is a first $V_{n+1} \in \mathcal{B}$ with diam $< 2^{-n-1}$ such that $K_1(S_{n+1} \cap V_{n+1})$ is non-meagre on U. Then $T_{n+1} := U \cap \mathrm{intcl}(K_1(S_{n+1} \cap V_{n+1}))$ is non-empty, and $K_1(S_{n+1} \cap V_{n+1})$ is heavy and dense on T_{n+1} (by Proposition L2). By regularity, we may pick in X an earliest non-empty $G_{n+1}' \subseteq \mathrm{cl} G_{n+1}' \subseteq T_{n+1} \subseteq U \subseteq G_n$. Summarizing, $K_1(S_{n+1} \cap V_{n+1})$ is heavy and dense in T_{n+1} and $G_{n+1}' \subseteq \mathrm{cl} G_{n+1}' \subseteq T_{n+1} \subseteq G_n$.

As $G_{n+1}' \subseteq G_n \subseteq T_n'$, (the last inclusion by (i)), by (iv) $K_2(S_n' \cap V_n')$ is non-meagre on G_{n+1}', so there is a minimal closed set S_{n+1}' in S_n' such that $K_1(S_{n+1}' \cap V)$ is non-meagre on G_{n+1}' for every $V \in \mathcal{B}$ meeting S_{n+1}'. So there is a first $V'_{n+1} \in \mathcal{B}$ with diam $< 2^{-n-1}$ such that $T'_{n+1} := G'_{n+1} \cap \mathrm{intcl}(K_2(S'_{n+1} \cap V'_{n+1}))$ is non-empty. Then $K_2(S'_{n+1} \cap V'_{n+1})$ is heavy and dense on T'_{n+1}. By regularity, we may pick in X a first non-empty $G_{n+1} \subseteq \mathrm{cl}G_{n+1} \subseteq T'_{n+1} \cap U_1 \cap ... U_{n+1} \subseteq T'_{n+1} \subseteq G'_{n+1}$.

We define $\pi(U_0, ..., U_n)$ to be G_{n+1}.

Taking $F_n := \mathrm{cl}G_n$ (and noting that $G_n \subseteq \mathrm{cl}G_n \subseteq G_{n-1}$, ,by (i)) we may apply Theorem 2 to deduce that

$$(A_1 \cap A_2) \cap \bigcap\nolimits_n U_n \supseteq (A_1 \cap A_2) \cap \bigcap\nolimits_n G_n = (A_1 \cap A_2) \cap \bigcap\nolimits_n F_n \neq \emptyset.$$

Since U_0 was arbitrary, we see that $A_1 \cap A_2$ is dense. Furthermore, for U_0 arbitrary non-empty and W_n dense open sets, we may let β play U_0 and then $U_n := W_1 \cap ... \cap W_n \cap G_n$ for $n \geq 1$. Then

$$(A_1 \cap A_2) \cap U_0 \cap \bigcap\nolimits_n W_n \supseteq (A_1 \cap A_2) \cap \bigcap\nolimits_n U_n = (A_1 \cap A_2) \cap \bigcap\nolimits_n F_n \neq \emptyset.$$

So $A_1 \cap A_2 \cap U_0$ meets any dense $\mathcal{G}_\delta$ so is non-meagre. Thus $A_1 \cap A_2$ is heavy.

The measure analogue follows by Lemma $2'$. $\square$

3.2 Luzin's Separation Theorem from Analytic Cantor

The last result replaced the σ-ideal $\mathcal{M}$ by $\mathcal{N}$ and exploited implicitly the notion of $\mathcal{N}$-heavy. Here we use the *omission* σ-ideal $\mathcal{I}_A$, which comprises those sets C which may be covered by a Borel set B missing the set A. So A' is $\mathcal{I}_A$*-heavy* if for every open set U meeting A' there is no Borel set $B \in \mathcal{I}_A$ with $A' \cap U \subseteq B$. This observation leads to a new approach to Luzin's Theorem below. A Borel set B is said to *Borel-separate* A_1 from A_2 if $A_1 \subseteq B$ and $B \cap A_2 = 0$. So, if this separation cannot be done, then A_1 is not in $\mathcal{I}_A$ for $A = A_2$. For analytic sets non-separation translates to heaviness; for $\mathcal{K}$-analytic sets this may be salvaged to a localization asserting that $A_1 \cap G \notin \mathcal{I}_A$ for some open set G, and this suffices to construct a 'strategy' in the sense of the last section. (This follows an idea exploited in [OT].) Below, for simplicity, we take the Borel sets to be those generated from the open sets. When convenient, we refer to A_2 simply as A. Our first result is only for the purposes of comparison.

Omission Localization Lemma A (Analytic). *In a metric space for A_1 analytic, the $\mathcal{I}_A$-light part of A_1, A_L, is in $\mathcal{I}_A$. If $A_1 \notin \mathcal{I}_A$, then $A_1 \backslash A_L$ is analytic and $\mathcal{I}_A$-heavy.*

Proof. Taking $\mathcal{W} := \{G \in \mathcal{G}(X) : G \cap A_1 \in \mathcal{I}_A\}$, the light part is $A_L := A_1 \cap \bigcup \mathcal{W}$. By Lemma 1(i) A_L is analytic, so Lindelöf; and so, as $\mathcal{W}$ covers A_L, there is a countable sequence of sets G_n in $\mathcal{W}$ covering A_L. We may assume each G_n meets A_L. So, as $G_n \cap A_1 \in \mathcal{I}_A$, there is a Borel set B_n such that $G_n \cap A_1 \subseteq B_n$ and $B_n \cap A_2 = 0$. But $\bigcup_n B_n$ is Borel, covers A_L and is disjoint from A_2, so $A_L \in \mathcal{I}_A$.

Now $A_H := A_1 \backslash A_L = A_1 \backslash \bigcup \mathcal{W}$ is closed relative to A_1 and so analytic (cf. [Jay-Rog] Th. 2.5.3). If x is in the light part of A_H, then for some open neighbourhood G of x, we have $G \cap A_1 = (G \cap A_H) \cup (G \cap A_L) \in \mathcal{I}_A$, and so $x \in A_L$, a contradiction. □

Omission Localization Lemma L (Lindelöf). *For A_1 Lindelöf, if $A_1 \notin \mathcal{I}_A$ (i.e. A_1 is disjoint from A and not Borel separated from A), then $A_1 \cap G$ is not in $\mathcal{I}_A$ for some open G meeting A_1.*

Proof. If not, then for each open G meeting A_1, since $G \cap A_1 \in \mathcal{I}_A$, there is a Borel set C_G with $G \cap A_1 \subseteq C_G$ and $C_G \cap A_2 = 0$. Then $B_G := G \cap C_G$ is a Borel subset of G such that $G \cap A_1 \subseteq B_G$ and $B_G \cap A_2 = 0$. So $\{G \in \mathcal{G}(X) : \emptyset \neq G \cap A_1 \in \mathcal{I}_A\} \supseteq \{G \in \mathcal{G}(X) : G \cap A_1 \neq \emptyset\}$, and the latter covers A_1, so since A_1 is Lindelöf there is a countable sequence of such sets G_n covering A_1. But $\bigcup_n B_{G_n}$ is Borel, covers A_1 and is disjoint from A_2, a contradiction. □

Luzin's Theorem (cf. [Jay-Rog], Th. 3.3.1)**.** *In a regular Hausdorff space, if A_1, A_2 are disjoint $\mathcal{K}$-analytic sets, then there exists a Borel set B containing A_1 and disjoint from A_2.*

Proof. We use the notation of §1.1. Let $A_1 = K_1(I)$ and $A_2 := K_2(I)$ be disjoint analytic subsets of a regular space X. Suppose the analytic sets are not Borel separated. Then, by Omission Lemma L, there is G_1 open such that $G_1 \cap A_1$ is not Borel-separated from $G_1 \cap A_2$. Also, for some i_1 and j_1 the sets $K_1(i_1) \cap G_1$ and $K_2(j_1) \cap G_1$ are not Borel separated. Indeed, if each $K_1(n)$ was separated from A_2 by B_n then $\bigcup_n B_n$ covers A_1 and is disjoint from A_2. Likewise, if each B_n covers A_1 and is disjoint from $K_2(n)$, then $\bigcap_n B_n$ covers A_1 and is disjoint from A_2.

By regularity, using sets G with cl$G \subseteq G_1$ we may likewise find $G_2 \subseteq$cl$G_2 \subseteq G_1$ such that $K_1(i_1) \cap G_2$ and $K_2(j_1) \cap G_2$ are not Borel separated. Also, for some i_2 and j_2 the sets $K_1(i_1, i_2) \cap G_2$ and $K_2(j_1, j_2) \cap G_2$ are not Borel separated. Continuing inductively, we select a nested sequence G_n and two points i and j in I such that $K_1(i|n) \cap G_n$ and $K_2(j|n) \cap G_n$ are not Borel separated. So neither is empty, as G_n is Borel; so for $F_n :=$clG_n we have that $K_1(i|n) \cap F_n$ and $K_2(j|n) \cap F_n$ are both non-empty. We may apply Theorem 2$'$, taking $A_{2n} := A_2$ and $A_{2n+1} := A_1$, to deduce that $\bigcap F_n \cap K_1(i) \cap K_2(j) \neq \emptyset$, a contradiction to the disjointness of A_1 and A_2. □

Remarks. 1. Separation theorems of Novikov-type (swelling analytic sets without common part to Borel sets without common part, see [DJRO]) may be proved by the same technique.

2. Less 'separable' versions of Localization Lemma L may readily be developed (appropriate to 'non-separable' descriptive theory, cf. [Han-92], or [St2]). In a metric space, recall that the *extended Borel sets* form the smallest σ-algebra generated from closed sets using countable intersection and σ-discrete unions. So with 'Borel' interpreted throughout as 'extended Borel', assume in Lemma L above in place of the Lindelöf property that A_1 has a σ-discrete base for its topology, say $\mathcal{B} = \bigcup\{\mathcal{B}_n : n \in \mathbb{N}\}$ with each $\mathcal{B}_n$ discrete. To prove the lemma still holds, suppose otherwise. Then, for each $G \in \mathcal{B}_n$ with $G \cap A_1 \in \mathcal{I}_A$, there is a Borel set $C = C_G$ with $G \cap A_1 \subseteq C$ and $C \cap A_2 = 0$. Then $B_G := G \cap C_G$ is a Borel subset of G such that $G \cap A_1 \subseteq B_G$ and $B_G \cap A_2 = 0$. Take $B_n := \bigcup\{B_G : G \in \mathcal{B}_n\}$ and, as $\mathcal{B}_n$ is discrete, so is $\{B_G : G \in \mathcal{B}_n\}$; so B_n is (extended) Borel, $\bigcup_n B_n$ covers A_1 and omits A, as before.

3. Louveau's separation theorem in [Lou] is a further example.

4. There is a similar appeal to heavy analytic sets employed in Harring-

ton's proof of Silver's Theorem on Π_1^1-equivalence relations: see [Mil] Ch. 30 and [MK] §9.4.

5. For use of other ideals see e.g. Solecki [Sol1] (where a number of classical theorems, asserting that a 'large' analytic set contains a 'large' compact subset, are deduced) For a closely related result we note *Solecki's dichotomy* that, for $\mathcal{I}$ a family of closed sets (in any Polish space) and $\mathcal{I}_{\text{ext}}$ the sets covered by a countable union of sets in $\mathcal{I}$, any analytic set A is either in $\mathcal{I}_{\text{ext}}$, (i.e. is $\mathcal{I}_{\text{ext}}$-light), or contains a $\mathcal{G}_\delta$ set not in $\mathcal{I}_{\text{ext}}$ – cf. Proposition L3 in §3. (See also [Sol2] for further applications of dichotomy.)

4 Fine Topology Analytic Baire Theorem

In §3 the Baire theorem was deduced for refinement topologies via weak bases, under the assumption that their elements were associated with some *global* family of certain analytic sets of the ground space. Here we again study the Baire theorem in the setting of two topologies, one metric and the other a *fine* topology, i.e. a refinement of the metric one, called therefore a *submetrizable* topology. Now we are concerned with refinements defined by *local* properties, but our aim is still to exploit the van Mill argument (via the local connections between the ground topology and its refinement).

To state the result (Theorem 6 below) we begin with notation and definitions and, for motivation and context, we pause to consider some local properties of refinements that are of natural interest.

For (X, d) a separable metric space, $\mathcal{T}_d$, as before, denotes the topology generated by d, $\mathcal{B}$ a countable base for $\mathcal{T}_d$ and $\mathcal{A}(\mathcal{T}_d)$ the analytic subsets of X under $\mathcal{T}_d$. Below, $\mathcal{T} \supseteq \mathcal{T}_d$ denotes the refinement under study (rather than $\mathcal{T}'$). We have in mind for $\mathcal{T}$ particularly $\mathcal{GH}$, the Gandy-Harrington topology of §3; we note in passing the case $\mathcal{T} = \mathcal{T}_d$.

Unsubscripted notation, such as cl (for closure) is with respect to $\mathcal{T}_d$ and subscripted, such as $\mathrm{cl}_{\mathcal{T}}$, is with respect to the $\mathcal{T}$-topology; thus int and $\mathrm{int}_{\mathcal{T}}$ denote the respective interiors, while 'open' or 'closed' means open or closed in $\mathcal{T}_d$. Note that $\mathrm{cl}_{\mathcal{T}} A \subseteq \mathrm{cl} A$. We refer below to a σ-ideal $\mathcal{I}$.

Definition ($\mathcal{I}$-topology). Say that the topology $\mathcal{T}$ is an $\mathcal{I}$-*topology* if for each $A \in \mathcal{A}(\mathcal{T}_d)\backslash\mathcal{I}$ the set $\mathrm{cl}_{\mathcal{T}}A$ has non-empty $\mathcal{T}$-interior. The topologies $\mathcal{T}_d$, $\mathcal{D}$ and also $\mathcal{GH}$ (§3 above) are $\mathcal{I}$-topologies for $\mathcal{I} = \mathcal{M}$, $\mathcal{N}$, $\{\emptyset\}$, respectively.

Definitions ($\mathcal{I}$-inclusion). 1. Write $A \subseteq_{\mathcal{I}} B$ if $A\backslash B \in \mathcal{I}$, and say that A is $\mathcal{I}$-*almost included* in B. Note that $\subseteq_{\mathcal{I}}$ is a transitive relation ($A \subseteq_{\mathcal{I}} B$ and $B \subseteq_{\mathcal{I}} C$ implies $A \subseteq_{\mathcal{I}} C$ as $A\backslash C \subseteq (A\backslash B) \cup (B\backslash C)$).

This definition is motivated by Proposition L3.

2. Following [ChMa], say that U is the *quasi-interior* of A (relative to $\mathcal{I}$), and write $U =$q-int(A), if U is the largest open set in X with $U \subseteq_{\mathcal{I}} A$. So (q-int$A)\backslash A \in \mathcal{I}$ and q-int$A \in \mathcal{T}_d$.

3. Say that $\mathcal{T}$ is an $\mathcal{I}$-*refinement* of $\mathcal{T}_d$ on X if for any $\mathcal{T}$-closed A and $G \in \mathcal{T}_d$, $A \subseteq G$ implies that q-int$(A) \subseteq G$ and further if $\mathrm{int}_{\mathcal{T}}A$ is non-empty, then so is q-int(A).

4.1 Examples C: Localization refinements

The fine topologies on $\mathbb{R}$ below have all played a significant role in analysis and topology.

1.(a) *Ideal-neglecting topologies.* If the σ-ideal is translation-invariant and $\mathcal{I}$ satisfies the localization property (cf. §1), then a topology that neglects members of $\mathcal{I}$ may be defined so that G is open in the *ideal-neglecting topology*, in brief i-open, iff G takes the form $U\backslash Z$ with U arbitrary open and any $Z \in \mathcal{I}$; see [LMZ] p. 25.

(b) The case $\mathcal{I} = \mathcal{N}$, studied in [Sch], gives a topology $\mathcal{T}$ with $\mathcal{T}_d \subseteq \mathcal{T} \subseteq \mathcal{D}$. The topology $\mathcal{T}$ may be called the *essential topology* (by analogy with the term 'essential supremum'; cf. [BOst-KCC] §2). Compare [Mar1] Th. 6.

2. *Two O'Malley topologies*:

(a) In connection with approximate differentiablity, we noted in §2.1 O'Malley's r-topology $\mathcal{R} \subseteq \mathcal{D}$; here one has $\mathcal{M}(\mathcal{R}) = \mathcal{M} = \mathcal{M}(\mathcal{T}_d)$ and $\mathrm{cl}A\backslash\mathrm{cl}_r A \in \mathcal{M}$ (cf. [O, Cor. 3.3]).

(b) O'Malley [O] also considers the a.e. topology $\mathcal{AE} \subseteq \mathcal{R} \subseteq \mathcal{D}$; for $D \in \mathcal{AE}$ one demands that $|D| = |\mathrm{int}D|$. So here $\mathrm{int}D \subseteq_{\mathcal{N}} D$.

3. In the absence of a translation-invariant ideal, translation-invariant fine topologies on $\mathbb{R}$ may be generated by first defining convergence at the origin (relative to a refined neighbourhood base at the origin). Then translation is used to define convergence at other locations.

(a) *Scheinberg's maximal topology* $\mathcal{U} \supseteq \mathcal{D}$ (see [Sch]) has the following lifting property: any bounded measurable real-valued function f is equal a.e. to a unique function $\tilde{f}$ continuous relative to $\mathcal{U}$. (His modification refers to an ultrafilter of measurable sets extending the filter $\mathcal{D}_0 := \{D \in \mathcal{D} : 0 \in \mathcal{D}\}$.)

(b) *Wilczyński topologies:* Wilczyński (see e.g. [Wil]) modifies the Lebesgue-density topology in several ways and also transfers the modification to a '*Baire-category density*' topology by defining the density of a set E at 0 in terms of the indicator functions $1_{nE\cap[-1,1]}$ (i.e. indicators of the traces around the origin of the dilations nE). Then convergence in a chosen (functional) mode of convergence is demanded.

(c) For a fixed null sequence $z_n \to 0$ one may declare S open iff $s + z_n \in S$ for any $s \in S$ and all but a finite number of indices n. This is related to the Kestelman-Borwein-Ditor Theorem (see e.g. [BOst-LBII]).

4. The *Kunen line* (for which see [JKR], or [JR] §4.1) is a refinement topology $\mathcal{T}$ of $\mathcal{T}_d$ on $\mathbb{R}$ with the property that a $\mathcal{T}$-open set differs from an open set by a countable set. For $\mathcal{I}$ the countable sets, $\mathcal{T}$ is an $\mathcal{I}$-topology.

Remarks. 1. The topologies satisfy various axioms separation, sometimes stopping short of normality. We recall that a topology $\mathcal{T}$ has the *Luzin-Menchoff property* relative to $\mathcal{T}_d$ (cf. [LMZ] Th. 3.11) if for each closed set H and $\mathcal{T}$-open U there is a $\mathcal{T}$-open V with $H \subseteq V \subseteq \mathrm{cl}V \subseteq U$. Recalling that $\mathcal{F}(\mathcal{T}_d) \subseteq \mathcal{F}(\mathcal{T})$, this may be viewed as a restricted 'normality' property in that some $\mathcal{T}$-closed sets (those that are $\mathcal{T}_d$-closed) may be separated from any disjoint $\mathcal{T}$-closed set ($X\backslash U$, say) by $\mathcal{T}$-open sets (V and $X\backslash \mathrm{cl}V$). Thus the density topology $\mathcal{D}$ and also $\mathcal{R}$ has the Luzin-Menchoff property (see [GNN]).

2. For $\mathcal{I} = \mathcal{M}$ and X a Baire space (i.e. everywhere non-meagre), or respectively for $I = \mathcal{N}$ and X everywhere *locally positive* in measure, note that q-int$(A) \subseteq$ cl(A). Indeed, otherwise if $x \in$q-int$(A) \setminus$ cl(A), then x

has a $\mathcal{T}_d$ open nhd $U \subseteq$cl$(U)\subseteq$q-int(A) with cl$(U)\cap A = \emptyset$, contradicting (q-int$A)\backslash A \in \mathcal{I}$, as respectively either U is non-meagre or $|\text{cl}(U)| > 0$.

3. For $\mathcal{I} = \mathcal{M}$, X a Baire space (i.e. everywhere non-meagre), and A closed, write $A = U\backslash N \cup M$ with N, M in $\mathcal{I}$, $N \subseteq U$ and $M \cap U = \emptyset$. Then one has q-int$(A)= U$ (since $U \subseteq_{\mathcal{I}} A$ and $A\backslash U \in \mathcal{I}$). So for A closed, if $A \subseteq G$ with G open in X, then $U =$q-int$(A) \subseteq G$. Moreover, if A is non-meagre, then q-int(A) is non-empty.

4. For $\mathcal{I} = \mathcal{N}$, the condition $\text{cl}_{\mathcal{D}}(M)\backslash\text{int}_{\mathcal{D}}(M) \in \mathcal{N}$ for all M is equivalent to the density theorem holding (see [LMZ] p. 172).

5. Note that for $A = [0,1] \cap \mathbb{Q}$ one has cl$A = [0,1]$ and $\text{cl}_{\mathcal{D}}A = A$, as $[0,1]\backslash\mathbb{Q}$ is open in $\mathcal{D}$. Thus in $\mathcal{D}$, unlike in $\mathcal{R}$, the two closures, are not equal modulo $\mathcal{N}$ (despite being metrically close – for U open with $\text{cl}_{\mathcal{D}}A \subset U$, one has cl$A \subset U$).

4.2 Fine Analytic Baire Theorems

We now modify the proof of [vM] Prop 2.2 to the context of fine topologies, replacing $\mathcal{M}$ by $\mathcal{I}$. This enables us to conduct completeness arguments modulo $\mathcal{I}$. Note that, for $\mathcal{T}_d$ Baire and $U_n = G_n\backslash Z_n$ with $Z_n \in \mathcal{I}$ and G_n open and dense, it does not follow from $\bigcap_n G_n \neq \emptyset$ that $\bigcap_n U_n \neq \emptyset$. Matters may be different when $\mathcal{I}$ has the Borel envelope property; then, for Borel $B \supseteq \bigcap_n (X\backslash Z_n)$, we have $\bigcap_n U_n \supseteq \bigcap_n G_n \cap B \neq \emptyset$, provided each $G_n \cap B$ is analytic and $\mathcal{I}$-heavy. So the first assertion of Theorems 6 and 6$'$ is stronger than its second.

Theorem 6 (Analytic Heavy Sets Theorem). *For $\mathcal{T}$ an $\mathcal{I}$-topology which is an $\mathcal{I}$-refinement of $\mathcal{T}_d$, if A and B are under $\mathcal{T}_d$ analytic, dense and $\mathcal{I}$-heavy, then $A \cap B$ is non-empty. If in addition $\mathcal{I}$ has the Borel envelope property, then $A \cap B$ is dense and $\mathcal{I}$-heavy.*

Proof. Take $A = \alpha(P)$ and $B = \beta(Q)$, where P, Q are Polish and α, β are continuous. By Proposition L1 we may assume additionally that for every non-empty open set V in P the set $\alpha(V)$ is not in $\mathcal{I}$, and the same for β. (That is, passage to the minimal subspaces of Proposition L1 does not remove density.) This additional property of $\mathcal{I}$-faithfulness ($\mathcal{I}$-irreducibility) implies that cl$\alpha(V)$ has non-empty $\mathcal{T}$ interior, as $\mathcal{T}$ is an $\mathcal{I}$-topology. For convenience, we assume that the diameter of P and Q is at most 1.

Assume inductively that there are open sets U_n, V_n, W_n (respectively, in P, Q, X) of diameters less than 2^{-n} such that

$$\mathrm{cl}\alpha(U_n) \subseteq_{\mathcal{I}} \mathrm{cl}\beta(V_n) \subseteq W_n \text{ and } \mathrm{cl}\alpha(U_n) \subseteq W_n,$$

with $U_0 = P, V_0 = Q$, $W_0 = X$. For the base step of the induction, note that cl$\alpha(U_0)$ =cl$\beta(V_0) = X = W_0$, as A and B are dense.

Passing to the inductive step, by the additional property of $\mathcal{I}$-faithfulness ($\mathcal{I}$-irreducibility), $\mathrm{cl}_{\mathcal{T}}(\alpha(U_n))$ has non-empty $\mathcal{T}$-interior, and so also non-empty quasi-interior, as $\mathcal{T}$ is an $\mathcal{I}$-refinement. Pick W non-empty open in $\mathcal{T}_d$ with diam(W)$< 2^{-n-1}$ and

$$W \subseteq \mathrm{cl}(W) \subseteq \text{q-int}(\mathrm{cl}_{\mathcal{T}}(\alpha(U_n))) \subseteq_{\mathcal{I}} \mathrm{cl}_{\mathcal{T}}(\alpha(U_n)) \subseteq \mathrm{cl}(\alpha(U_n)).$$

Now $W \cap \beta(V_n) \neq \emptyset$, as $\subseteq_{\mathcal{I}}$ is transitive (from transitivity one has that $W \subseteq_{\mathcal{I}}cl\beta(V_n)$). So $\beta^{-1}(W) \cap V_n \neq \emptyset$ (and is open, as β is continuous). By regularity of Q and $\mathcal{T}_d$, and continuity of β, we may pick a neighbourhood F in Q of diameter less that 2^{-n-1} such that both cl$F \subseteq V_n$ and cl$\beta(F) \subseteq W$.

Now $\mathrm{cl}_{\mathcal{T}}\beta(F)$ has non-empty $\mathcal{T}$-interior, and hence non-empty quasi-interior, again as $\mathcal{T}$ is an $\mathcal{I}$-refinement. The latter is contained in W, as $\mathrm{cl}_{\mathcal{T}}\beta(F) \subseteq$cl$\beta(F) \subseteq W$ and cl$\beta(F)$ is closed. Now let W' be any non-empty open set in $\mathcal{T}_d$ with

$$W' \subseteq \mathrm{cl}(W') \subseteq \text{q-int}(\mathrm{cl}_{\mathcal{T}}(\beta(F))) \subseteq W.$$

Again $W' \cap \alpha(U_n) \neq \emptyset$, as $\subseteq_{\mathcal{I}}$ is transitive (from transitivity we have $W' \subseteq_{\mathcal{I}}cl\beta(F) \subseteq W \subseteq_{\mathcal{I}}cl\alpha(U_n)$). So $\alpha^{-1}(W') \cap U_n \neq \emptyset$. By regularity of P and $\mathcal{T}_d$ and continuity of α, we may pick a neighbourhood E in P of diameter less that 2^{-n-1} such that both cl$E \subseteq U_n$ and cl$\alpha(E) \subseteq W' \subseteq W$.

Taking $U_{n+1} = E$, $V_{n+1} = F$ and $W_{n+1} = W$ yields

$$\mathrm{cl}\alpha(U_{n+1}) \subseteq W' \subseteq_{\mathcal{I}} \mathrm{cl}\beta(F) = \mathrm{cl}\beta(V_{n+1}) \subseteq W = W_{n+1},$$

which gives the inductive step, i.e. that

$$\text{cl}\alpha(U_{n+1}) \subseteq_{\mathcal{I}} \text{cl}\beta(V_{n+1}) \subseteq W_{n+1} \text{ and } \text{cl}\alpha(U_{n+1}) \subseteq W_{n+1}.$$

Let $\{p\} = \bigcap_n U_n \in P$ and $\{q\} = \bigcap_n V_n \in Q$; then $\alpha(p)$ and $\beta(q) \in W_n$. But $\text{diam}(W_n) \to 0$ as $n \to \infty$, so $\alpha(p) = \beta(q) \in A \cap B$, as asserted.

Now assume $\mathcal{I}$ has the Borel envelope property (Definition 3, §1). For analytic set A, B that are dense and $\mathcal{I}$-heavy and for any non-empty open set G, the set $A \cap G$ is analytic (as G is an $\mathcal{F}_\sigma$ in the $\mathcal{T}_d$ topology), and dense in G; also, since G is open, A is $\mathcal{I}$-heavy in G. Likewise $B \cap G$ is analytic, dense and $\mathcal{I}$-heavy in G. So working in G rather than X, we have $A \cap B \cap G$ non-empty. So $A \cap B$ is dense in X. If $A \cap B$ is not $\mathcal{I}$-heavy, then $A \cap B \cap G$ is in $\mathcal{I}$ for some open G. By the Borel envelope assumption, there is a Borel set C in $\mathcal{I}$ such that $A \cap B \cap G \subseteq C$; then both $G \cap A \backslash C$ and $G \cap B \backslash C$ are analytic, dense and $\mathcal{I}$-heavy (by Lemma 4). But then $(G \cap A \backslash C) \cap (G \cap B \backslash C)$ is non-empty, and yet

$$\emptyset = (G \cap A \cap B) \backslash C = (G \cap A \backslash C) \cap (G \cap B \backslash C),$$

a contradiction. So $A \cap B$ is $\mathcal{I}$-heavy. □

The same argument, but using an induction which visits each analytic set infinitely often, gives:

Theorem 6′ (Fine Analytic Baire Theorem). *Let $\mathcal{T}$ be an $\mathcal{I}$-refinement of $\mathcal{T}_d$. If A_n are (under $\mathcal{T}_d$) analytic subsets of X that are dense and $\mathcal{I}$-heavy, then $\bigcap_n A_n$ is non-empty. If in addition $\mathcal{I}$ has the Borel envelope property, then $\bigcap_n A_n$ is dense and $\mathcal{I}$-heavy.*

Remarks. 1. In [Ost-LBIII] we develop a similar inductive argument in which the intersection $\bigcap_n U_n$ above avoids a countable number of meagre sets needing to be neglected. (The latter arise in the course of the induction.)

2. The proof above is again game-theoretic in character and so Theorem 6′ may be rephrased in the language of the multiple-target Choquet game to

conclude with the assertion that $\prod_n A_n$ is weakly α-favourable for sets A_n that are analytic and dense and $\mathcal{I}$-heavy.

Essentially the same argument as for Theorem 3 proves the following final result, which relies on the following

Observation. For $\mathcal{B} = \{B_n\}$ a countable base, if for each n the set A_n is non-empty, analytic and $\mathcal{I}$-heavy on B_n, then $A := \bigcup_n A_n$ is analytic, dense and $\mathcal{I}$-heavy. (For this conclusion, it suffices that $\mathcal{B}$ is a *weak base:* for each non-empty open V we need only a non-empty $B \in \mathcal{B}$ with $B \subseteq V$.)

Theorem 3′ (Fine Generalized Gandy-Harrington Theorem). *For $\mathcal{T}$ with a countable weak base $\mathcal{B}$, $\mathcal{H} \subseteq \mathcal{A}(\mathcal{T})$ a weak base for $\mathcal{T}'$ and $\mathcal{A}(\mathcal{T})$ closed under countable intersections of $\mathcal{I}$-dense members, the topology $\mathcal{G}_{\mathcal{H}}$ is Baire.*

In particular, in the setting of Theorem 6, if $\mathcal{I}$ has the Borel envelope property then $\mathcal{G}_{\mathcal{H}}$ is Baire.

Proof. For each n, let G_n be dense and open in $\mathcal{G}_{\mathcal{H}}$. Choose $H_{nm} \subseteq G_n \cap B_m$ with $\emptyset \neq H_{nm} \in \mathcal{H}$ and $B_m \in \mathcal{B}$ (possible, since G_n, being $\mathcal{G}_{\mathcal{H}}$-dense, meets every open set in $\mathcal{B}$, and $\mathcal{H}$ is a weak base). Then $G_n' := \bigcup_m H_{nm}$ is $\mathcal{T}$-dense, and being locally $\mathcal{I}$-heavy $\mathcal{T}$-everywhere, is $\mathcal{I}$-heavy. As each G_n' is analytic, by Theorem 6′, $\emptyset \neq \bigcap_n G_n' \subseteq \bigcap_n G_n$.

For V an arbitrary non-empty open set in $\mathcal{G}_{\mathcal{H}}$, as the set $G_n \cap V$ is $\mathcal{G}_{\mathcal{H}}$-dense on V, we may repeat the argument replacing $\mathcal{B}$ above by $\mathcal{B}_V := \{B \in \mathcal{B} : B \cap V \neq \emptyset\}$, and reading $V \cap G_n$ for G_n, to deduce that the intersection $V \cap \bigcap_n G_n$ is non-empty. So $\bigcap_n G_n$ is $\mathcal{G}_{\mathcal{H}}$-dense. $\square$

5 Concluding comments and remarks

1. p-space analogues.

We recall the definition of p-spaces, modifying that given in [Gr] to fit in with our vocabulary of $\mathcal{K}$-analytic spaces above (and the notation of §1.1), and to clarify the standing of our results. (They are sometimes taken to be completely regular, cf. [Gr] p. 441.)

Definitions 5.1. Define the product space $J = \kappa^{\mathbb{N}}$ for κ a cardinal with the discrete topology. In a topological space X, an open determining system $G := \langle G(j|n)\rangle$ is called an (open) *sieve* if for each $n \in \mathbb{N}$ and $j \in J$ the open family $\mathcal{G}_n(j|n) := \{G(j|n, j_{n+1}) : j_{n+1} \in \kappa\}$ is a covering of $G(j|n)$. Call X a *p-space* if for each j the set $G(j)$ is compact and the mapping $j \to G(j) := \bigcap_n \mathrm{cl}G(j|n)$ is upper-semicontinuous for those j with $G(j) \neq \emptyset$ (in the sense that $\emptyset \neq G(j) \subseteq U$, with U open, implies that $\bigcap_{n\leq m} \mathrm{cl}G(j|n) \subseteq U$ for some m). The sieve is called *complete* if $\bigcap\{\mathrm{cl}F : F \in \mathcal{F}\} \neq \emptyset$ for each filterbase $\mathcal{F}$ provided there is $j \in J$ such that for each n there is $F \in \mathcal{F}$ with $F \subseteq G(j|n)$. A space is *sieve-complete* if it has a complete sieve. Such spaces are also called monotonically Čech-complete in [ChCN].

For later use, amend the preceeding definition of a complete sieve by requiring j as above to satisfy $G(j) \neq \emptyset$, and then say that the sieve is *weakly complete*. Also call the sieve $G = \langle G(j|n)\rangle$ *Souslin-like* if upper-semicontinuity arises at *all* j. (These are the 'pseudo-complete' sieves of [Mich86].)

In certain circumstances, for instance in the metric context (as noted in §1), the sets $K(i|n)$ in the representation of a $\mathcal{K}$-analytic set may be expanded (given a $\mathcal{G}$-circumscribed representation) to yield an open sieve. Notwithstanding this, the proofs of Theorems 4 and 5 may be re-interpreted as referring to general Souslin-like sieves, and in turn these results may also be viewed as being about certain kinds of p-spaces.

We note in particular that X is called *pointwise countably complete* relative to the sequence of open covers $\mathcal{U}_n$ if, whenever F_n is decreasing and non-empty and, for each n, $F_n \subseteq U_n \in \mathcal{U}_n$ for sets U_n with $\bigcap_n U_n \neq \emptyset$, then $\bigcap_n \mathrm{cl}F_n \neq \emptyset$. (This corresponds to an Inclusion version of the Analytic Cantor Theorem, Th. 1_{C}.) So p-spaces are pointwise countably complete relative to $\mathcal{G}_n$ and exhibit a Cantor-Theorem property. The need to guarantee a non-empty intersection $\bigcap U_n$ or $\bigcap G(j|n)$ above may be satisfied when working with *strong* Choquet-style games as defined in §3.1.

Conversely, from a Souslin perspective, a p-space is obtained by passing to a *weak* Souslin-like sieve, in which upper-semicontinuity is not demanded when $G(i) = \emptyset$. This was the point of view taken in [Ost-2], studying p-spaces that are metacompact or paracompact (or even Lindelöf).

Definitions 5.2 ([Frol-60], Th.4.5 & 4.6; cf. [CP] p. 500). The (open) determining system G is an *almost sieve* if each $\mathcal{G}_n$ above is an *almost cover* (i.e. the union of each $\mathcal{G}_n(j|n)$ is dense in $G(j|n)$); X is *almost complete* if it has a complete almost sieve.

Motivated by the p-space definition, Cao and Piotrowski [CP] define a game MP(X) by altering the definition of winning rule in the strong Choquet game (§3.1) to require that either (i) $\bigcap_n U_n = \emptyset$, or (ii) $K(U) = \bigcap U_n$ is compact for $U = \langle U_n \rangle$ and, for every open W with $K(U) \subseteq W$, there is n with $U_n \subseteq W$. The existence of a winning strategy in MP(X) is equivalent to the existence of an (open) weakly complete sieve. An X with such a sieve is called a monotonic p-space, or briefly an *mp-space*. Thus X is an *mp*-space iff MP(X) is α-favourable.

An *almost mp-space* is defined similarly, replacing sieves by (open, weakly complete) almost sieves. We quote:

Theorem CP (Cao and Piotrowski, [CP] Th. 3.3). *A regular space X is almost complete iff X is weakly α-favourable and X is almost mp.*

Remarks. 1. See [ChCN] Ex 2.9 for an example of a locally completely metrizable space that is metacompact, but not Čech-complete.

2. Every almost complete, completely regular space contains a Čech-complete, dense $\mathcal{G}_\delta$-subspace. (Compare Proposition L3 of §3.)

3. The localization technique is captured abstractly by coverings $\mathcal{V}$ of a space A that are *exhaustive*, in the sense that every non-empty $S \subseteq A$ contains a non-empty relatively open subset of the form $S \cap V$ with $V \in \mathcal{V}$. This notion is introduced and used in [Mich86] to define a complete exhaustive sieve (in the manner above) and to derive from it a notion of α-favourability.

4. Topsøe [Top] in 1981 characterizes sieve-complete spaces X by a variant of the Choquet game in which the winning rule requires that each net eventually in every U_n has a cluster point in X (see also [CP] Th. 2.2). Michael [Mich86] Th. 7.4 characterizes 'strictly exhaustive' sieves that are pseudo-complete (i.e. Souslin-like) in terms of weak α-favourability.

5. We recall from §2.2 the remark that [Kech] Th. 25.18 shows that for any sequence of analytic subsets A_n of a Polish topology $\mathcal{T}_d$ there exists a second-countable, strongly Choquet (as above) refinement $\mathcal{T}$ under which each A_n is open and that [Kech] Ex. 25.19 cites Becker's result ([Bec] Th. 1.2) that for any second-countable, strong Choquet topology $\mathcal{T}$ extending a Polish topology $\mathcal{T}_d$, the members of $\mathcal{T}$ are analytic in the sense of $\mathcal{T}_d$. This nicely straddles Theorem CP and weak α-favourability of heavy analytic sets (Th. 4), as these are almost complete.

6. Arhangel'ski introduced p-spaces in [Arh1] (see also [Arh2] and [Gr]); they share some of the features of the later generalizations of [WW]. Using the terminology based on the Souslin operation, like that above (rather than 'sieves'), Ostaszewski established the connection between $\mathcal{K}$-analyticity and p-spaces (at the Analytic Sets Conference in 1978 ([Rog]), see [Ost-2]); this includes preservation theorems for countable products, which are of interest here in relation to the multi-target Choquet games of §3.1. The sieve perspective was studied again more recently by E. Michael (see [Mich86] 1986, [Mich91] 1991).

2. Effros Theorem.

Recently, van Mill [vM] extended what is now known as the Effros Open Mapping Principle ([Eff] Th. 2.1, or [Anc], for the more recent view; for other proofs see [ChMa] and one attributed to Becker in [Kech-T], and also [Ost-ACE] and the short proof in [Ost-E]) for the setting of an analytic (rather than Polish) group A acting on a non-meagre metric space X in a separately continuous fashion (i.e., $(a, x) \to a(x)$ is separately continuous in a and in x). The power of the analytic approach is such that the proof in fact establishes the following stronger result. We need some definitions.

Definitions. 1. For X an *algebraic* group, say that $|| \cdot || : X \to \mathbb{R}_+$ is a *group-norm* ([BOst-N]) if the following properties hold:

(i) Subadditivity (Triangle inequality): $||xy|| \leq ||x|| + ||y||$;

(ii) Positivity: $||x|| > 0$ for $x \neq e$;

(iii) Inversion (Symmetry): $||x^{-1}|| = ||x||$.

2. The right and left *induced norm topologies* are given by the right and left invariant metrics: $d_R^X(x, y) := ||xy^{-1}||$ and $d_L^X(x, y) := ||x^{-1}y|| = d_R^X(x^{-1}, y^{-1})$.

3. An algebraic group G with identity e acts *transitively* on a space X if for each x, y in X there is g in X such that $g(x) = y$. If G as a set is given a topology, then this action is separately continuous if $(g, x) \to g(x)$ is continuous *separately* in g and in x. (On this point see Comment 5.3 below.) The group acts *micro-transitively* on X if for any open neighbourhood U of e in G and any $x \in X$ the set $\{h(x) : h \in U\}$ is a neighbourhood of x. The following result was proved by van Mill ([vM]) for G an analytic *topological* group; his proof in fact gives:

Analytic Effros Open Mapping Principle. *For G an analytic **normed group** acting transitively and separately continuously on a separable metrizable space X: if X is non-meagre, then G acts micro-transitively on X.*

Indeed, van Mill notes that he uses (i) separately continuous action (see the final page of his proof), (ii) the existence of a sequence of symmetric neighbourhoods U_n of the identity with $U_{n+1} \subseteq U_{n+1}^2 \subseteq U_n$, and (iii) $U_1 = G$ (see the first page of his proof). By [BOst-N, Th. 2.19′] (the Birkhoff-Kakutani Normability Theorem) van Mill's conditions under (ii) specify a normed group, whereas condition (iii) may be arranged by switching to the equivalent norm $||x||_1 := \max\{||x||, 1\}$ and then taking $U_n := \{x : ||x||_1 < 2^{-n}\}$.

The normed group result is of interest, as some naturally occurring normed groups are not complete (see Charatonik et Maćkowiak [ChMa] for Borel groups that are not complete). For further generalizations see [Ost-E].

3. Bouziad's Theorems.

A second example on the use of analyticity from the group context is

Bouziad's Theorem [Bou1], that an algebraic group endowed with a topology under which it is Baire, Čech-analytic and has separately continuous multiplication (so a semitopological group) is a topological group. A further result in similar spirit is on separately continuous actions, again due to Bouziad (see [Bou2]): a left-topological group (i.e. with continuity only of left shifts, as happens in normed groups) that is Baire, completely regular, analytic and with separately continuous action on an analytic space has continuous action; this covers the context of the Effros Principle in the preceeding Comment 5.2. (More is true: in particular, we may replace analytic spaces here by p-spaces.) In this connection one should note that, by a theorem of Loy [Loy] and Hoffman-Jørgensen [HJ], an analytic topological group that is Baire is a Polish space; for a generalization see [Ost-Joint].

4. The direct Baire property of Levi.

The paradigm for Proposition L1 in §3 is Levi's argument in [Levi], which hinges on the *direct Baire property* of an analytic space A (open subsets of A, being analytic, map to Baire sets). This is equivalent to continuous maps, suitably restricted, becoming open maps; we recall this briefly in a simplified context. For $f : I \to A$ continuous and with $\mathcal{B}$ a countable base for I, noting that for each $V \in \mathcal{B}$ the image $f(V)$ is analytic, we put $f(V) = (U_V \backslash M_V) \cup N_V$ with M_V, N_v meagre and U_V open. Consider $A_0 = \bigcup\{M_V : V \in \mathcal{B}\} \cup \bigcup\{N_V : V \in \mathcal{B}\}$, which is meagre (so assume also an $\mathcal{F}_\sigma$), and $A' = A \backslash A_0$. Take $S = f^{-1}(A')$. For V in $\mathcal{B}$, $f(V \cap S) = U_V \cap A'$, and so $f|S : S \to A'$ is open, and induces a countable base for A'. In particular, A' is metrizable, in addition to being a dense absolute-$\mathcal{G}_\delta$ in A (by Hausdorff's theorem on the metric preservation of completeness). For a non-separable analogue, see [Ost-AB].

4. Multiple targets and product spaces.

It is not true in general that a product of Baire spaces is Baire (see [Oxt1] §4). By contrast, results about the multiple target situations are implied by results in the individual co-ordinate spaces: see [Wh] Th. 3(4), or [Choq] 7.12(iv).

5. The heart of a set.

The inductive processes throughout have relied on the 'heart' of a heavy set H, some such set as $\mathrm{intcl}(H)$ or the quasi-interior. In this connection see [Jay-Rog] Lemma 2.9.1 for the *Baire envelope* of a set (especially the proof which refers to the light and heavy parts of a set), and antithetically [Kech] Th. 8.29 for the properties of the set $U(A) := \bigcup\{G : G \text{ open and } A \cap G$ co-meagre on $G\}$. Of course a Baire non-meagre set is 'locally co-meagre' on its quasi-interior. Compare also [St1].

6. Regular variation, measure-category duality.

The normed group version of the Effros theorem is of interest in the theory of *topological regular variatio*n, see e.g. [Ost-knit] and [BOst-TRI], especially the need for an analytic version to cover the observed absences of completeness. (The non-complete examples, already cited at the end of Remark 2 from [ChMa], are normed groups.) It is this that led to the present study both of analyticity and of fine topologies. In particular, the density topology in a bitopological approach allows *topological* regular variation to unify the classically established 'dual' theory [BGT] in that it has a measure-theoretic formulation on the one hand and a topological (or Baire) formulation on the other. This is done in two ways. In [BOst-LBI] the theory is developed in a larger class of functions, giving a common generalization of the measurable and Baire-property cases. In [BOst-LBII] a topological embedding theorem, and the resulting shift-compactness, are the unifying common features of measurable sets and sets with the Baire property. The latter approach was instrumental in [BOst-KCC] in eliciting new measure results from known category results.

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